Penrose-Newmann formalism :Special relativity in null coordinates

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Summary The tetrad formalism, used in general relativity, is based on an orthonormal basis to define the local spacetime which is that of special relativity. Special relativity is generally defined in Minkowski coordinates (t, x, y, z), in a Galilean frame of reference, where the paths of light rays are also defined.

Breaking with this point of view, Penrose and Newmann [6], by a new paradigm, proposed to reverse the approach by using null coordinates, defined on a basis made of set of 4 null vectors where only 3 are linearly independent, adapted to light paths, as a reference in special relativity instead of those of Minkowski which are a remnant of the heritage of Newtonian mechanics. This is motivated by several (correlated) reasons.

First, It is the existence of a "maximum" velocity (defined with space and time concepts), associated with the velocity of light that implies an hyperbolic structure of spacetime. One may wonder how such sentence should make sense, as this "'maximum velocity"' destroys the physical nature of space and time in this sentence! Conversely, as null geodesics are physical spacetime entities, null geodesics, unlike timelike and spacelike worldlines, are consistent with the description of spacetime. moreover, for philosophical consistency a fundamental physical concept (spacetime) should not result of combination of non physical concepts (space and time) even though this would be compatible in dimensional analysis. Sspace and time, non physical concepts, may result of arbitrary slicing of spacetime but this result is arbitrary. This explains why null geodesics, physical spacetime entities, are more efficient for describing the physical spacetime. But, let us stress that it is the **geometrical structure of null geodesics** which represents the physical phenomenon, as photons (light) are just spacetimelike objects co-moving on null geodesics

Second, as the number of different Galilean frames of reference is infinite and as they are all equivalent, the choice of one of them is arbitrary, while the fact that the celerity of light is the same in all, makes it unique, therefore not arbitrary.

Third, and maybe the most important, the physics in general relativity, described in this way is on the one hand more homogeneous as all coordinates are null, this meaning that null coordinates are genuine spacetimelike coordinates in accordance with the real nature of general relativity, instead of timelike or spacelike coordinates which are the concepts of the Newtonian mechanics, and on the other hand, more efficient because is supported by only 3 degree of freedom instead of four. Moreover, this approach involves products of first order partial derivatives in the metric instead of a quadratic form of second order partial derivatives, this usually simplifying the equations and allowing its orientation, something useful in solutions including horizons which is not possible with a pure quadratic form. The epistemological implications of this will be developed in chapter 4.

We claim that, by this choice, as more information about the the phenomenology is included in the formalism, this will simplify the calculations and would enlighten the understanding of the structural properties of this spacetime As an example, we show that in null coordinates, as defined by the Newmann-Penrose (NP) formalism [6], the formalism is simpler. We will propose a phenomenological interpretation of this result.

1 Minkowski's metric

Original Minkowski's metric, in t, x, y, z coordinates, is recalled in equation (1) below.

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

These equations can be integrated, this yields:

$$s^2 = -t^2 + x^2 + y^2 + z^2 \tag{2}$$

The special relativity only describes inertial frames which are the geodesics of this spacetime. The ten symmetries of the Minkowski's spacetime are defined by the Poincaré group. This group describes the set of

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transformations which is made of three boosts, which are "‘rotations in the planes *t,x* or *t,y* or *t,z*"’, three space rotations in the planes *x, y* or *x, z* or *y, z* and four translations in the directions *t, x, y, z*. These transformations keep unchanged the values of ds^2 and s^2.

In analytic geometry, the *t, x, y, z* original coordinates are transformed in other coordinates *t', x', y', z'* by this operation. This can be written:

$$x'^{\mu} = M^{\mu}_{\nu} x^{\nu} \tag{3}$$

where M^{μ}_{ν}, also noted shortly M, in this document, is a 4x4 matrix and ν and μ vary from 0 to 3.

This implies that the original Minkowski's metric tensor $\eta_{\mu\nu}$,

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 1

is transformed into an other metric's tensor noted $\eta'_{\mu\nu}$ in *t', x', y', z'* coordinates, which will be defined further.

2 Transformation matrix and null coordinates for the metric

In analytic geometry, for leaving ds^2 invariant, the coordinates x^{μ} and x'^{μ} must satisfy :

$$ds^2 = dx^{\mu} \eta_{\mu\nu} dx^{\nu} = dx'^{\mu} \eta'_{\mu\nu} dx'^{\mu} \tag{4}$$

This relation implies that the set of all compliant transformation matrices M, in symbolic notations, must satisfy:

$$\eta = M^T \eta' M \tag{5}$$

Which can be developed as:

$$\eta_{\mu\nu} = M^{\rho}_{\mu} \eta'_{\rho\lambda} M^{\lambda}_{\nu} \tag{6}$$

Where M^{ρ}_{μ} is the transposed matrix M^{λ}_{ν}.[1]

3 The null coordinates in the Newmann-Penrose formalism [2]

Per this formalism, and renaming the coordinates (*t', x', y', z'*) respectively (U, V, W, W^*), following Newman and Penrose, we will define:

$$x'^0 = t' = U = \frac{t+x}{\sqrt{2}}, x'^1 = x' = V = \frac{t-x}{\sqrt{2}}, x'^2 = y' = W = \frac{y+i.z}{\sqrt{2}}, x'^3 = z' = W^* = \frac{y-i.z}{\sqrt{2}} \tag{7}$$

The factor $\sqrt{2}$ is for the normalization of the ds^2.[3]

The metric becomes:

$$ds^2 = -2dU.dV + 2dW.dW^* \tag{8}$$

whose metric tensor is:

[1] See [1] p. 12-15 for a full demonstration.

[2] In this formalism, the Einstein equation requires the resolution of a system of 30 first order partial differential equations, in place of the usual system of ten second order highly non linear partial differential equations. These 30 equations are almost linear, this facilitating their numerical resolution and the most important number of parameters allows to set more easily some conditions

[3] In the Minkowski form, as there are four degrees of freedom, an orthonormal vectors basis is made of 4 unitary and orthogonal independent vectors. In null coordinates, we have one constraint on these vectors: they must be null, therefore there is only three degrees of freedom left. Let us notice that a null vector is orthogonal to itself so there are only three orthogonal independent null vectors. As we need four vectors for the basis, this explains the complex form of the last 2 space vectors.

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure 2

The transformation matrix M^{μ}_{ν} is:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix}$$

Figure 3

multiplied by a factor $2^{-1/2}$

It is straightforward to check that all coordinates (U, V, W, W^*) are null and that the vectors tangent to the coordinates $(\partial_\mu)^\nu$, for μ and ν defined in null coordinates, below,

$$\begin{aligned} (\partial_U)^\mu &= [1,\ 0,\ 0,\ 0] \\ (\partial_V)^\mu &= [0,\ 1,\ 0,\ 0] \\ (\partial_W)^\mu &= [0,\ 0,\ 1,\ 0] \\ (\partial_W*)^\mu &= [0,\ 0,\ 0,\ 1] \end{aligned}$$

Figure 4

are null, this meaning that:

$$(\partial_\rho)^\mu \eta'_{\mu\nu} (\partial_\rho)^\nu = 0$$

for $\rho = U, V, W, W*$ in null coordinates; In addition these null vectors satisfy the requirements of the Newman-Penrose formalism i.e :

$$\begin{aligned} (\partial_U)^\mu \eta'_{\mu\nu} (\partial_V)^\nu &= -1 \\ (\partial_W)^\mu \eta'_{\mu\nu} (\partial_{W^*})^\nu &= 1 \\ (\partial_U)^\mu \eta'_{\mu\nu} (\partial_W)^\nu &= 0 \\ (\partial_U)^\mu \eta'_{\mu\nu} (\partial_{W^*})^\nu &= 0 \\ (\partial_V)^\mu \eta'_{\mu\nu} (\partial_W)^\nu &= 0 \\ (\partial_V)^\mu \eta'_{\mu\nu} (\partial_{W^*})^\nu &= 0 \end{aligned}$$

figure 5

We see that this null vectors basis, unlike in the Minkowski formalism, is not orthonormal.[4]

4 Some epistemological implications of this approach

In the introduction, we listed some properties of the null coordinates formalism. A key point is that in this formalism there are only three degrees of freedom, this is since in a basis of null vectors [5] there are only 3 linearly independent "'orthogonal"' vectors, because a null vector is orthogonal to itself. Does this allow to describe the same spacetime than the Minkowski's one ? In other words, can we describe a four degrees of freedom phenomenology by a three degrees of freedom phenomenology? Let us stress that the constraint bound to null vectors applies on the basis and its associated coordinates, not on the structure of spacetime which is constrained by the metric (the ds^2).

[4] In Minkowski form, the vectors basis $(\partial_\alpha)^\beta$ satisfies $(\partial_\rho)^\mu \eta_{\mu\nu} (\partial_\lambda)^\nu = \delta^\rho_\lambda$

[5] As we use the analytic geometry for describing the geometry over the manifold describing the spacetime, this basis is associated to the null coordinates. Let us recall that in general relativity the metric tensor (the ds^2) does not depend on the coordinates.

The transformation rules between the Minkowski's basis of vectors and the basis of null vectors, where for compatibility we introduced two complex conjugate null coordinates W and W* [6], that we deduce in previous chapter, ensures that we did not introduce any restriction. In this null basis, by using the null coordinates we can describe any timelike, spacelike or null worldline, exactly as in the Minkowski's basis and coordinates. Often, starting from a Minkowskian description a conversion into null basis is performed for simplifying intermediate calculations, the result of which can be converted back into a Minkowski's basis, without any loss of information. Therefore, we are entitled to say: "' Yes, the two descriptions are fully equivalent.

A first epistemological major conclusion would be that, as in Minkowskian description 4 degrees of freedom are requested and as in the null description we need only 3, considering that information needed is the same, this means that the null formalism includes more information than the Minkowski's one! In other words, null geodesics contain more information [7] about the spacetime nature than the archaic representation by a 3 D space and a 1 D time, inherited from the Newtonian's physics.

Moreover, the Newman-Penrose formalism (null formalism) reveals the spinor structure of the hyperbolic spacetime of the special relativity [8], which seems interesting for linking special relativity and quantum mechanics[9].

5 Example of comparison of transformation matrix in Minkowski's metric and in null metric

For a boost, in the plane (t, x), of parameter φ such that $v/c = tanh(\varphi)$ and a spatial rotation of an angle of θ in the plane (y, z), in the Minkowski's metric, the transformation matrix M can be written,[10]:

$$\begin{pmatrix} \cosh(\varphi) & -\sinh(\varphi) & 0 & 0 \\ -\sinh(\varphi) & \cosh(\varphi) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Figure 6

While in the Newman-Penrose formalism, for the same transformations we get:[11]

$$\begin{pmatrix} \exp -\varphi & 0 & 0 & 0 \\ 0 & \exp \varphi & 0 & 0 \\ 0 & 0 & \exp -i\theta & 0 \\ 0 & 0 & 0 & \exp i\theta \end{pmatrix}$$

Figure 7

This result exhibits an interesting similarity with that obtained by the analysis in group theory. The group SO(3,1) of symmetries, boost and rotations of four-vectors, of the Lorentz spacetime (excluding translations) is given in equation (III-64) of [4]. Moreover, the construction of the relation between SO(3,1) and SL(2,C) [12], described in [4] equation (III-73) relies on a 2x2 Hermitian matrix whose components are:
$t + z, x - iy, x + iy, t - z$, which is something similar to the definition of coordinates U, V, W, W^* in equation (7).

Even though, the aim of the Newman-Penrose formalism was different, per these similarities with the group theory analysis, aimed to grasp all the symmetries of a structure, it is not surprising that, as Chandrasekhar pointed out, that the Newman-Penrose's formalism is also more efficient in grasping the symmetries, described by the group SO(3,1), of the spacetime than the Minkowski's formalism. The spinor representation of the Newman-Penrose formalism, would undoubtedly be also similar to that used in the group theory analysis where

[6]This does not increase the number of degree of freedom as they are not independent.

[7]Null geodesics rule the conformal structure of spacetime and therefore the causality.

[8]An interesting point is about the signature of the metric which is (-, +, +, +) in Minkowski's formalism and would be (0, 0, 0, 0) in null formalism ? This looks inappropriate, maybe that the concept of signature of a metric should be generalized in order to include other formalisms than the Minkowski's one.

[9]The Dirac equation and the QFT widely use a spinor formalism.

[10][1] p. 13-14

[11]See details of the demonstration in appendix 1

[12]SL(2,C), where 2 is the dimension and where C denotes the complex numbers set, is the two dimensional representation associated to spinors, ruled by the same Lie algebra than SO(3,1) but whose topology is simplex

it is the group SL(2,C) which will be in use.

At last but not at least, the choice of null coordinates is also comforted by the fact that this choice simplifies the form of some equations. Therefore we are entitled to think that this choice will reflect better the physical nature of the phenomenology.

6 Representation by the relativistic Doppler phenomenology

As our analysis will suggest to propose a description by frequency shift instead of space length and time interval, it is worth to analyze what would be the results of such analysis by using the relativistic Doppler phenomenology. We will see that this will similar to the Newman-Penrose formalism. See appendix 2

7 The physical phenomenology revealed in null coordinates ?

7.1 Introduction

Usually, one uses a formalism with a timelike coordinate and three spacelike coordinates which looks more compatible with our concepts of time and space. As such inertial frame (a geodesic) can be synchronized, this allows to recover a familiar "'pseudo - Newtonian"'phenomenology where time and space look to be independent. But we know that this synchronization is only valid within a frame and that, between frames, synchronization, relying on universal simultaneity capability, which is not assumed in relativity, is not possible.

The use of null coordinates in special relativity is not a common choice, as its physical interpretation may look unclear. This null tetrad formalism was introduced by Newman and Penrose, in 1962 in general relativity, mainly for the study of black holes. In this document, we argue that this formalism would be also very helpful for understanding the structure of the special relativity spacetime, even in this simpler context. The comments of S.Chandrasekhar in [3], listed further in this chapter, will enlighten the purpose of such formalism.

7.2 Properties of the Minkowski formalism

In the Minkowski formalism, the only entities which are universal are the value of the ds^2 which is also an affine parameter of any curve in spacetime and the celerity of light, which is the same for all frames [13].

The parametrization of the geodesic path, allowing to locate a point on this spacetime curve, is usually performed by **using the proper time of the observer** attached to this geodesic, noted τ such that $c^2 d\tau^2 = -ds^2$, which is also the dynamic parameter used in geodesic equations, for timelike frames. The infinity of inertial frames will use it, except that of the light because $ds^2 = 0$. The nature of the frame of light is different of the nature of the infinity of all other frames and usually one say that we cannot define a frame for light. This is illustrated by the fact that the light relative celerity is the same in all frames notwithstanding with their relative velocity, a non-intuitive property.

In this Minkowski's approach, the, arbitrary among an infinity, reference frame is the **timelike inertial frame where our observer is located**. All other inertial frames, which comply with the symmetry of the Minkowski's spacetime are related to our observer's frame by the values of the boost and/or rotation. Let us point out that we consider only inertial frames, but non inertial frames are possible even though the special relativity does not describe them. But there are extensions of the special relativity which describe these non-inertial frames in a spacetime which is no longer the Minkowski spacetime.

An other difference is that the light always follows geodesics paths. But we see that whether the relative velocity of light is the same for all inertial frames, it is not the case for its frequency, as measured by the observers on different frames. As the frequency of a photon is bound to its energy ($E = h\nu$) and as for the light the affine parameter will be its four-momentum directly bound to the frequency, even though the velocity of light is the same for all observers as they observe different frequencies they observe different geodesics of the light.

[13]This second invariant, settled as a postulate by Einstein, is in fact a consequence of the relativity principle, which implies a celerity invariant, but does not give explicitly its value but implicitly it is that of light

In this representation, using space coordinates and time coordinates, the phenomenology, including that of light, is described from the point of view of the observer on a physical timelike inertial frame. In other words, the other frames and the light are seen and described from the point of view of the observer.

7.3 Properties of the Newman-Penrose formalism

In [3] in its introduction of this formalism p. 40, Chandrasekhar declared:

"'The Newman-Penrose formalism is a tetrad formalism [14] with a special choice of the basis vectors. The choice that is made is a tetrad of null vectors l, n, m, m^*, [15] of which l and n are real and m and m^* are complex conjugates of one another. The novelty of this formalism, when it was first proposed by Newman and Penrose in 1962, was precisely in their choice of a null basis.[16]

It was a departure from the choice of an orthonormal basis which was customary till then. The underlying motivation for the choice of a null basis was Penrose's strong belief that the essential element of a spacetime is its light-cone structure which make possible the introduction of a spinor basis. And it will appear that the light-cone structure of space-times of the black-hole solutions of general relativity is exactly of the kind that makes the Newman-Penrose formalism most effective for grasping the inherent symmetries of these space-times and revealing their analytical richness.

But it may be stated already that the special adaptability of the Newman-Penrose formalism to the black-hole solutions of general relativity derives from "'type D"' character [17] and the Goldberg-Sachs theorem, matters which will be considered later in this chapter (§9) and subsequently."'

In addition to the interesting comments of S.Chandrasekhar on this formalism, we may also point out that:
The interest of this point of view is that as the velocity of light is a common parameter to all physical observers on theirs inertial frames, one is led to consider it as a privileged reference. This is a unique structural property that no other frame shares as there is no privileged frame in all timelike inertial frames. Such property suggests that light should be a conceptual key for understanding the structural geometry and topology of the spacetime. This is an invite to study this representation. As stressed by S. Chandrasekhar, the specificity and unique character of the light is comforted by the fact that, in relativity, as the finite and constant velocity of light rules the causality, it plays a structural role in the physical phenomenology. We know that in relativity universal time and space no longer exist, but luckily the causality is generally well defined at least in special relativity.

7.4 Frequency as the affine parameter on null geodesics

When $ds^2 = 0$, the geodesic is a null geodesic, i.e a geodesic of light. Unlike timelike worldlines, which can be geodesic or no geodesic, the light is always geodesic. The only parameter which may change is the frequency, that we may use as affine parameter for giving a scale, on the geodesic. Therefore instead of describing the phenomenology in terms of space length and time intervals, we will describe it in terms of frequencies, which along with the celerity of light c, will provide the parameters of a kind of dual representation.

Using the light as a reference frame is often discarded, because one cannot use something derived from the ds^2 as affine parameter because it is null. But it is forgetting that by using, the light momentum, proportional to the frequency of the photon, we get a very interesting alternative. The description of a phenomenon such as the gravitational lensing would be simply described by such formalism by something similar to that of light in refractive medias. The frequency along the null path varies according to the variation of local gravitational field

[14] A tetrad formalism, also called a noncoordinate basis formalism, relies on local basis of vectors defined by tetrads in the coordinates: See [3] p. 33-40, for instance

[15] In this document $l = (\partial_U)^\mu, n = (\partial_V)^\nu, m = (\partial_W)^\mu, m^* = (\partial_{W^*})^\nu$

[16] Chandrasekhar note: Penrose was originally led to consider the introduction of a null basis from his interest in incorporating in general relativity spinor analysis in an essential way. We briefly consider this alternative approach to the Newman-Penrose formalism in Chapter 10 - 102. In this chapter, quite complex, the Newman-Penrose four-vector formalism will be "'translated"' into a spinor formalism for making it compatible with the Dirac equation formalism. For that, he will use the relation $V = Z^+ \sigma Z$, where V is a four vector, Z a spinor and σ a Pauli matrix, which allows to build four-vectors from spinors of spin 1/2. Then, he will substitute the spinors description of vectors into the vector description of the Newman-Penrose for developing his equations in spinors formalism

[17] The Petrov-Pirani classification defines several classes of space-time. Type D is relative to spacetime generated by a unique mass with some symmetries, featuring two double principal null directions see [3] p. 59-60

and therefore the count of periods of the frequency along the null geodesic, which is representative of the affine parameter is affected by the gravitational field. This provides an elegant clear and simple method for solving the problem.

7.5 The "‘light"’ has no velocity, in the usual sense, as this results from a structural property of spacetime

From the previous comments, we are led to assume that speaking of the velocity of light is an anthropomorphic point of view. The light, in fact any physical phenomenology exhibiting its properties, such electromagnetism waves or even gravitational waves, has no intrinsic velocity which would be the limit of velocity. It is merely a consequence of a geometrical property of spacetime: the existence of null geodesics where electromagnetic waves are represented. Therefore, it would be crucial to consider the physic from this point of view!

This means that, unlike in the Minkowski representation, in a null basis frame representation, the observer should be described from the point of view of the light. Even though, this may look quite an odd idea, as there is no observer riding photons, let us remember that, in his preliminary analysis, Einstein was wondering how the world would appear to him from this point of view.

7.6 The "‘null"’ geodesics fully define the geometry of the manifold representing the spacetime associated to the universe

There are a set of null geodesics defined in each point and in its vicinity of the manifold. We know that the geodesic deviation defining the deviation of a bundle of geodesics (congruence of geodesics), including null geodesics, is ruled by the Riemann tensor. Therefore, if the Riemann tensor is defined in each point of the geometry, null geodesics would fully define the geometry of the manifold representing the spacetime [18].

7.7 Comments on the interest of the null tetrad formalism

In the null formalism, the transformation of coordinates, resulting of a boost of argument φ in the plane *t, y* and by a rotation of angle θ in the plane *y, z*, is represented by a diagonal matrix instead of a non diagonal matrix.

This is the simplest representation that we can expect for such transformation. In addition, in a boost of argument φ the component associated to dU and dV are inverse as well as those for a rotation of angle θ. On the one hand this means that there are only 2 independent components in this 4x4 matrix and in the other hand this reveals a symmetry of the transformation. In a boost, the scale of coordinate U is modified by a factor $exp(-\varphi)$ and the scale of coordinate V by the inverse $\exp\varphi$), the product $dU.dV = dU'.dV'$ remaining constant. For the rotation, the coordinate W will be rotated by an angle $\exp(-i\theta)$ and W^* by an angle $\exp(i\theta)$, the product $dW.dW^* = dW'.dW^{*\prime}$ remaining constant. Moreover the form of these independent components are exponential functions which play a structural role of generators in mathematics.

This comforts the argument of Chandrasekhar stating that the null basis graps better the symmetry of the spacetime, as this was not obvious in the Minkowski approach. By describing, in the most symmetrical way, all transformations relative to the symmetries of the Minkowski spacetime this enlightens the nature of this spacetime.

7.8 Does the frequency representation open a way towards a relation of indeterminacy and consequently a possible quantification?

Formally, mathematically, the relation between the representation by continuous geometric values of space and time and the representation by waves is done by using Fourier transforms, direct or inverse according to the direction of the transformation. One of the well-known properties of this transformation is indeterminacy. For instance, the position of a photon and its frequency cannot be known simultaneously with infinite precision.

In relativity could we also use this property to open a way towards the quantification of the theory? .

[18] This is even more restrictive in vacuum, as the type of the geometry of spacetime is defined by the principal null direction classes of null geodesics, for instance see [3] p. 58-63 for the Petrov classification depending on the five Weyl complex scalars, where in the Newmann-Penrose formalism, in an adapted gauge, four of the five vanish. Each class is an infinite set of null geodesics. There are four classes of them, in a four-dimensional spacetime corresponding to roots of an equation of degree 4, but depending on the symmetries of the spacetime the number of different roots may be 4, 3, 2 or 1). Let us point out that these principal null geodesics were described by Elie Cartan as soon as in 1922 for the Schwarzshild spacetime, [2]. Let us emphasize that, when we say that the manifold is fully defined, this means that all the points, of coordinates (t, x, y, z), of the manifold are defined for any combination of the coordinates (t, x, x, y, z) valid on this manifold. This definition does not describe all possible (null, timelike, spacelike) geodesics. We do not care about non-geodesic worldlines because we consider only gravity.

7.9 The relation between ”‘action”’ and ”‘energy”’ enlightened by the NP formalism

The concept of ”‘action”’ in physics, the dimension of which is energy multiplied by a time, is not is not easy to conceive.Its importance, in the derivation of the laws in physics by using the least action principle also called extrema principle, shows that it represents something fundamental (fundamental properties of the system under investigation) even though its meaning and nature looks mysterious to our mind. The selection of light (more generally electromagnetic waves) as the fundamental parameter in the analysis in relativity, per the relation: $E = h.\nu$ shows how in this formalism the action is related to the energy a more well known concept. The amount of energy of a photon, quantum of a electromagnetic wave, is the action quantum h multiplied by the frequency ν. Something more understandable than in other formalisms. This would comfort the choice of a null basis for describing the relativity.

8 Conclusion

This non exhaustive survey was intended to demonstrate that an other approach than that of Minkowski was possible and constructive in special relativity for deriving very simply the transformations between frames which allow to grasp more deeply the symmetries of this spacetime. This analysis led us to assume an other point of view for explaining the phenomenology of spacetime in relativity where the observer is not the reference. It would also be interesting to develop a deeper reflexion on the structural similarities with the group theory analysis that we have only briefly evoked in this document.

9 Appendix 1 - Matrix transformation for boost or rotation

9.1 Boost

We have to select the t,x section of the matrix in figure 6. This 2x2 submatrix transforms coordinates t, x in coordinates t’, x’.

$$\begin{pmatrix} dt' & | & \cosh(\varphi) & -\sinh(\varphi) & | & dt \\ dx' & | & -\sinh(\varphi) & \cosh(\varphi) & | & dx \end{pmatrix}$$

By setting $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, this yields:

$$dt' = \cosh(\varphi)dt - \sinh(\varphi)\frac{dx}{c} = \gamma(dt - \frac{v}{c^2}.dx) \tag{9}$$

$$dx' = -\sinh(\varphi).c.dt + \cosh(\varphi)dx = \gamma(dx - v.dt) \tag{10}$$

Usually, for simplifying the equations, one set $c = 1$, here we reintroduce c for exhibiting the homogeneity of the equations.

In equation (9) and equation (10), the formula on the right is the original form of the Lorentz transformations. It is straightforward to verify that, per the definition of $\varphi = \arg\tanh(\frac{v}{c})$ i.e $\frac{v}{c} = \tanh(\varphi)$, the equations (9) and (10) are correct[19].

Equation (7) is valid in any frame, therefore:

$$dU' = \frac{dt' + dx'}{\sqrt{2}}, V' = \frac{dt' - dx'}{\sqrt{2}} \tag{11}$$

Inserting equation (9) and (10) in equation (11) yield:

$$dU' = (\frac{\cosh(\varphi)dt - \sinh(\varphi)dx - \sinh(\varphi)dt + \cosh(\varphi)dx}{\sqrt{2}}) = (\cosh(\varphi) - \sinh(\varphi))(\frac{dt + dx}{\sqrt{2}}) = (\exp(-\varphi))dU \tag{12}$$

$$dV' = \frac{\cosh(\varphi)dt - \sinh(\varphi)dx - (-\sinh(\varphi)dt + \cosh(\varphi)dx)}{\sqrt{2}} = (\cosh(\varphi) + \sinh(\varphi))(\frac{dt - dx}{\sqrt{2}}) = (\exp(\varphi))dV \tag{13}$$

[19] $\frac{v}{c} = \tanh(\varphi) \Rightarrow \frac{v^2}{c^2} = \tanh^2(\varphi) \Rightarrow 1 - \frac{v^2}{c^2} = 1 - \tanh^2(\varphi) \Rightarrow 1 - \frac{v^2}{c^2} = 1 - \frac{\sinh^2(\varphi)}{\cosh^2(\varphi)} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{\cosh^2(\varphi) - \sinh^2(\varphi)}{\cosh^2(\varphi)} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{\cosh^2(\varphi)} \Rightarrow \cosh(\varphi) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$. Therefore, as $\cosh^2(\varphi) - \sinh^2(varphi) = 1$, we deduce in the same way that $\sinh(\varphi) = \gamma(\frac{v}{c})$

The transformation which transforms *dU, dV* into *dU', dV'* by this boost is therefore performed by the sub-matrix 2x2 as described below:

$$\begin{pmatrix} dU' & | & \exp(-\varphi) & 0 & | & dU \\ dV' & | & 0 & \exp(\varphi) & | & dV \end{pmatrix}$$

9.2 Rotation

A similar demonstration will yield the submatrix which transform W, W^* into $W', W^{*\prime}$ as described below:

$$\begin{pmatrix} dW' & | & \exp(-i\theta) & 0 & | & dW \\ dW^{*\prime} & | & 0 & \exp(i\theta) & | & dW^* \end{pmatrix}$$

10 Appendix 2 - Relativistic Doppler effect

Boost of v

For a light source of frequency f_0 moving radially at velocity v, $-c < v < c$, from the observer frame, the Doppler effect is given by:

$$\frac{f}{f_0} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}} \tag{14}$$

As $\tanh(\varphi) = v/c$ where φ is the argument of the function such that $-\infty < \varphi < \infty$ we get:

$$\frac{f}{f_0} = \frac{\sqrt{1 - \tanh^2(\varphi)}}{1 - \tanh(\varphi)} = \frac{1}{\cosh(\varphi)(1 - \tanh(\varphi)} = \frac{1}{(\cosh(\varphi) - \sinh(\varphi)} = \exp(\varphi) \tag{15}$$

This result, which gives the ratio of frequencies in the Doppler phenomenology, is the same than that of the boost, in the null coordinates phenomenology, described in figure 7. This shows that the two approaches describe the same phenomenology.

11 Appendix 3 - Klein-Gordon and Dirac equations in null coordinates

11.1 Klein-Gordon equation

In Minskowski coordinates (t, x, y, z), the Klein-Gordon equation is usually written:

$$[-(\partial_t)^2 + (\partial_x)^2 + (\partial_y)^2 + (\partial_z)^2 + m^2]\psi(t, x, y, z) = 0 \tag{16}$$

it is well known that the solution for the wave function $\psi(t, x, y, z)$ of this equation is provided by a linear sum of plane waves.

In null coordinates (u, v, w, w*) this will become

$$[-2.(\partial_u).(\partial_v) + 2.(\partial_w)(\partial_w*) + m^2]\psi(u, v, w, w*) = 0 \tag{17}$$

One can notice that there are no longer second order partial derivatives, instead we have a product of first order partial derivative [20]. In addition, this form of operator is more compact (and symmetrical?) as it is a sum of 3 operators instead of 5.

The solution would, likely, be provided by a linear combination of exponential functions.

One may wonder whether there exists coordinates (U, V, W, W*) providing a more compact form such as:

$$[-a.(\partial_U).(\partial_V)(\partial_W)(\partial_W*) + m^2]\psi(u, v, w, w*) = 0 \tag{18}$$

[20]This is a general property of these coordinates for the equations of the general relativity. This is often helpful for solving these equations. For instance, in the Kerr solution, (rotating black holes), the Weyl tensor is fully defined only by one Weyl, (complex), scalar

11.2 The Dirac equation

For the Dirac equation one approach, in Minkowski coordinates (t, x, x, y, z), was to consider the following relation

$$(E-U)^2 = p_x^2 + p_y^2 + p_z^2 + m^2c^2 = (a_1.p_x + a_1.p_y + a_1.p_z + a_4.m.c)^2 \tag{19}$$

Where E is the energy, U the potential, p_i the space impulsion, associated at ∂_i operators and a_i are matrices to be defined as there is no solution with numbers.

In null coordinates (u, v,w ,w*) what would this become ?

To be continued

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