Painlevé's formalism eliminates Newtonian time

Summary

In 1921, in [6] Painlevé established a spatial, geometric and covariant formulation of Newtonian gravitation. This complemented his development of Einstein's general relativity (GR) to eliminate the singularity of the Schwarzschild solution on the horizon [5],[6] and permits a formal comparison between Newtonian theory and GR [3],[4]. This type of relativistic formalism has been long forgotten, and assumes that the Newtonian potential is a gauge applied to the Euclidean metric. From this formulation ignoring Newtonian time, we obtain a physical dynamic parameter that is dedicated to each observer in geodesic motion, and can be used, in place of Newtonian time, to fully describe the system. We show that this formalism is physically compatible with the classical one assuming certain relations between the dynamic parameters of the two formalisms[1]. This important piece of work also removes the "a priori" concept of absolute Newtonian time.

Introduction

Published in 1915 in Berlin during first World War, general relativity (GR) aroused little interest [2] until 1921, when the Nobel Prize was awarded to Einstein. At the "Académie des sciences", in France, Painlevé tried to improve the understanding of GR by comparing it with the classical mechanics in three articles [5], [6], [7].
The second article, compares the two theories, overcoming their differences due to the geometric nature of GR gravitation, by developing a geometric interpretation of the Newtonian gravity[2].

The geometric form of Painlevé Newtonian mechanics

After recalling the equation of classical mechanics, Painleve provides a covariant geometric description of the exterior field of the one body spherical solution in classical mechanics by stating:

"*As we can see, it follows that we can give to the theory of Newtonian gravitation the following formalism (according to the principle of least action [3]): The trajectories of the point P are geodesics of the following* ds^2

$$ds^2 = (U + h)\,(dx^2 + dy^2 + dz^2), \qquad (a)$$

where U is a function of x, y, z, which vanishes at infinity, whose ΔU *is null outside of the sphere S and is equal to a negative constant inside S ", h is a constant and* $U = M/r$. [4]

The Painlevé definition of a Euclidean conformal space

In contrast to the Newtonian's method where a mass produces a scalar potential in Euclidean space, Painlevé presents a geometrical approach similar to relativity, in which a mass curves the space geometry by multiplying it by a conformal factor.
The geometry becomes Euclidean conformal, hence is no longer flat. The Weyl tensor remains null, as both spaces have the same conformal (and causal) structure, but both the Ricci and Einstein tensors which model the gravity are not null. In general relativity, it is the opposite: The Ricci and Einstein tensors are null but not the Weyl tensor. This is commented in annex 2.

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[1] In his last article [7], he will generalize this formalism to general relativity for the "Schwarzschild"solution. We will show, in another article "Relativistic Schwarzschild's solution in Painlevé's spatial formalism", that, subject to a small modification of his article, this formalism allows to derive the solution.

[2] This has some limitations, in particular the phenomenon must not depend on time. This was prefigured by P. Appell [1], but for a different purpose (for a system in classical mechanics).

[3] In relativity the geodesic Lagrangian $L(x, dx/d\lambda) = \frac{1}{2}\, g_{\mu\nu}(x)\,(dx^\mu/d\lambda)(dx^\nu/d\lambda) = \frac{1}{2}\,(ds^2/d\lambda^2)$. See [9]

[4] Today we would rather define $U = -M/r$. But for the demonstration we will use his definition.

This illustrates that, even though both theories have similar geometric forms, they are different!

Equivalence of the geometric form of Painlevé with that of the Newtonian mechanics.

In the annex, we demonstrate the equivalence between the Newtonian and geometric formalism of classical mechanics. The Newtonian method involves two constants of motion, the angular momentum *C* and the total energy *E* such as:

$$C=r^2\frac{d\varphi}{dt}, \tag{1}$$

$$\frac{dr^2+r^2d\varphi^2}{2dt^2}=\frac{M}{r}+E \rightarrow dt^2=\frac{dr^2+r^2d\varphi^2}{2(\frac{M}{r}+E)}. \tag{2}$$

The first one is the second Kepler law, the second relation is the Hamiltonian of the system, conserved, as in the single spherically symmetrical body problem, it does not depend on time.

In Painlevé's geometric formalism defined by equation *(a)*, we also have two constants: *L* resulting from the existence of an angular Killing vector ***L***, as the metric does not depends on the coordinate φ, and the "4-velocity invariant" ($|U^\mu U_\mu| = -1$). The constant *h* is related to the total energy of the system. In such geometric formalism, *s* is the dynamic parameter.

As the spatial curves of the geodesics are plane (ellipses, parabolas, hyperbolas), we can set $\theta = \pi/2$ in both formalisms. The coordinates r, θ, φ are the same in the two formalisms, but as the dynamic parameters *t* and *s* are different, nothing implies that $C = L$ and $E = h$. Let's introduce a multiplicative constant *k,* which will be equal to *2* per the definition of the geodesic Lagrangian in relativity [9], in the "Painlevé metric" We will see that this will imply $C = L$ and $E = h$, therefore:

$$ds^2=k(\frac{M}{r}+h)[dr^2+r^2d\varphi^2], \tag{3}$$

$$L=k(\frac{M}{r}+h)r^2\frac{d\varphi}{ds}. \tag{4}$$

First step: Consistent spatial geodesic curves

Let's recall that in Newtonian formalism, these spatial geodesic curves are usually defined by[5]:

$$\frac{1}{r}=\frac{1+e.cos(f)}{p} \quad with \quad e=\sqrt{1+\frac{2C^2E}{M^2}}, \quad p=\frac{C^2}{M}, \quad f=\varphi-\omega. \tag{5}$$

In the annex we demonstrate that, in Painlevé's formalism, the spatial geodesic curve equation is:

$$\frac{1}{r}=\frac{1+e.cos(f)}{p} \quad with \quad e=\sqrt{1+\frac{4L^2h}{kM^2}}, \quad p=\frac{2L^2}{kM}, \quad f=\varphi-\omega, \tag{6}$$

identical to the equation *(5)* of the Newtonian mechanics for $k = 2$, ($C = L$ and $E = h$). Therefore:

$$ds^2=2(\frac{M}{r}+h)[dr^2+r^2d\varphi^2]. \tag{7}$$

From equations *(2)* and *(7)*, we deduce: $dt.ds = (dr^2 + r^2d\varphi^2) = dS^2$. *(8)*

[5] Where *M* is the Newtonian mass of the body generating the field, *C* is the angular momentum (constant) of its satellite of mass $m=1$, *E* is the total energy (constant) of the system in classical mechanics for this solution, *e* the "eccentricity" of the curve, *p* the "focal parameter" and *f* the "true anomaly". For $0 \le e < 1 \rightarrow -1 \le 2C^2E/M^2 < 0 \rightarrow E < 0$, we get an ellipse, a circle for $e = 0$: the system is bound. For $e = 1 \rightarrow E = 0$, we get a parabola: the system is critical. For $e > 1 \rightarrow E > 0$ we get a hyperbola: The system is unbound. First, we describe the case $C \neq 0$. . We will consider $C = 0$, later.

This equation reveals a formal duality between the dynamic parameters *t* and *s*.*Second step: consistency of the dynamics, duality between time and spacelike affine parameter*

For specifying the geodesic, we need also to check that motion on this spatial curve is the same. The motion along different segments *"X-Y"*[6] of equal length Δ*S, evaluated in Euclidean metric, on the spatial orbit* (the ellipse),will require unequal intervals of time Δ*t*, (Kepler, second law). The *length* Δ*s* of these "*X-Y*" segments, *evaluated in Painlevé's metric, (geodesic, in a conformal Euclidean space)* are unequal. For representing it, in Euclidean space, we can set milestones *P* on the orbit, marking the geodesic length Δ*s,* associated to *X-Y.* This length Δ*s,* is these of *R(φ), a curve, which is Painlevé's isometric image of the orbit,* solution of the differential equation (9-1) and *T(φ), the associated elapsed time is* defined by *(9-2),* both curves for *r,* defined by equation *(6)*[7].

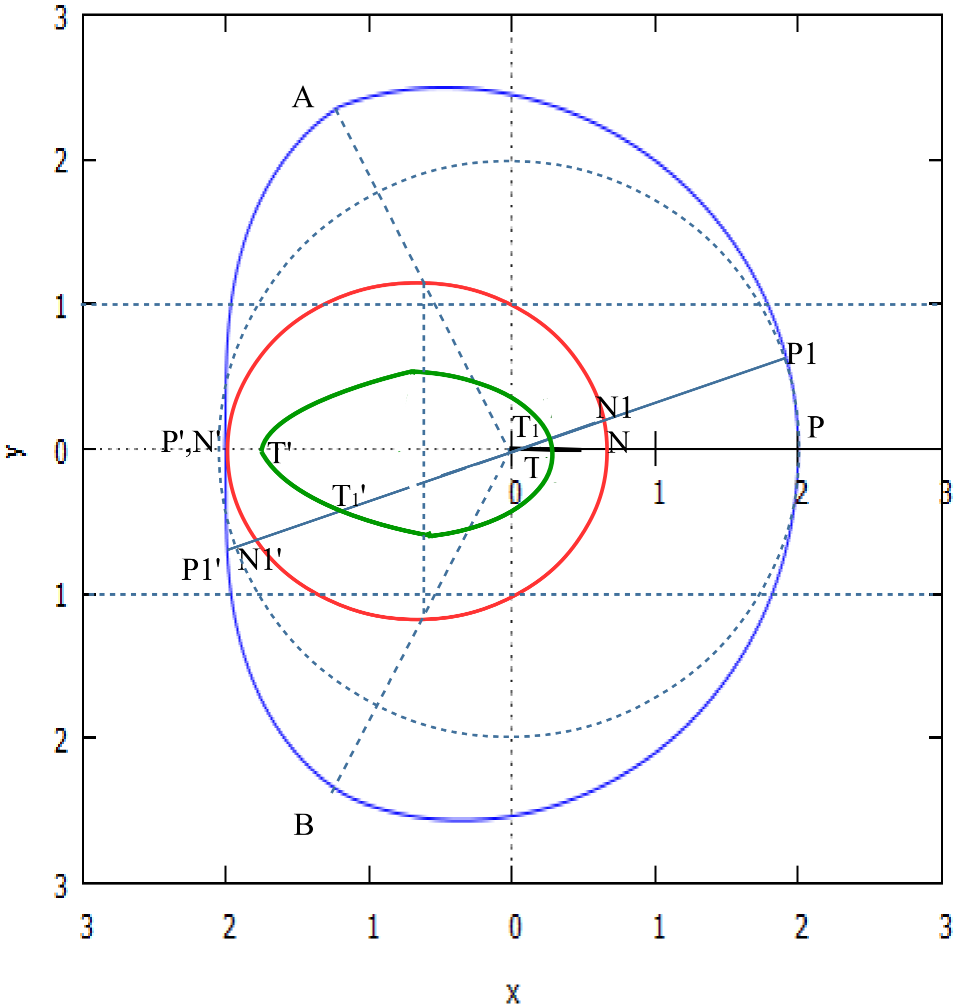

Fig. 1.The red ellipse of coordinates *(r, φ)* is an orbit (eccentricity = 0.5 in this example) in the Euclidean space. The blue curve *R(φ)* is, roughly, its conformal image, The green curve represents *T(φ).* Even though NN1 (elapsed time, TT_1) and N'N1' (elapsed time, $T'T'_1$) on the red curve are highly unequal, on the blue curve, PP1 and P'P1', their images in Painlevé's formalism, to be used as milestones for indicating their lengths in the conformal space, are equal.

[6] XY is a generic notation:P,P',P1,P1' for Painlevé's curve, N,N',N1,N1' for Newton's curve, T,T', T_1, T_1' for time's curve.

[7] R(φ) is defined by the differential equation $R'^2(\varphi) = f(R^2(\varphi), \varphi) \rightarrow R'(\varphi) = \pm\sqrt{f(R^2(\varphi), \varphi)}$ which define a family of curves depending on initial conditions for φ = 0. An approximate curve, computed by using numerical *rk* method, is drawn for $R(0) = 2 \rightarrow R'(0) = 0$. The blue curve is made of 4 parts , PA, AP', P'B, BP where we use alternatively: $R'(\varphi) = +\sqrt{f(R^2(\varphi),\varphi)}$ and $R'(\varphi) = -\sqrt{f(R^2(\varphi), \varphi)}$. The green curve defined for $T(0) = 1/4.5 \rightarrow T'(0) = 0$ is made of two parts (TT' and T'T) corresponding to $T'(\varphi) = +\sqrt{f(T^2(\varphi), \varphi)}$ and $T'(\varphi) = -\sqrt{f(T^2(\varphi), \varphi)}$.

$$[dR(\varphi)]^2+R(\varphi)^2*d\varphi^2=(\frac{4(\frac{M}{r}+h)^2}{L^2})r^4*d\varphi^2 \quad (9-1),\quad [dT(\varphi)]^2+T(\varphi)^2*d\varphi^2=\frac{r^4}{L^2}d\varphi^2 \quad (9-2)$$

Some properties of T(φ) and R(φ) curves.

Even though the equation *R(φ)* defines a family of an infinity of isometric curves, depending on initial value *R(0)*, the value of *R(π)* will be the same in all of these curves.This generates an application where a curve generates an infinity of other curves. We selected *R(0) = 2,* which implies *R'(0) = 0.* From *φ = 0* up to *φ = 2π,* we discover four extremum of *R(φ)*, at *φ = 0, φ =2π/3, φ = π, φ =4π/3* implying $R'(\varphi)=+\sqrt{f(R^2(\varphi),\varphi)}$ to change into $R'(\varphi)=-\sqrt{f(R^2(\varphi),\varphi)}$ and, conversely, this implies that the curve is made of four segments joined together. Even though the curve is continuous, the derivative may be not continuous, this depending on initial conditions. Same kind of remarks apply to *T(φ)*, but with only two extremum (*φ = 0* and *φ = π*). We selected *T(0) = 1/4.5* which implies *T'(0) = 0*. We see that for *φ = π, T'(π) ≠ 0,* this may look quite odd, but we have to keep in mind that the relation is defined between affine parameters of the curves.

What about the Painlevé original equation?

Equation *(7)* is more convenient for comparing the two formalisms. Does this mean that Painlevé was wrong? The annex, eq. *(1-2)* shows that for $k = 1$, $2L^2 = C^2$, that is consistent with eq. *(6)*. The Painlevé equation describes the same phenomenology but under a different parameterization.

Unification of the formalism of Newtonian mechanics

This geometric formalism unifies the hybrid representation of classical mechanics (two equations) in a single formalism, while promoting the spacelike affine parameter of the geodesic, as dynamic parameter of the geodesic observer, in place of the absolute time of the classical mechanics. The duality between *s* and *t* ensures the consistency and the equivalence of the two formalisms.

Painlevé uses the gauge freedom in his proposal

In classical mechanics, the length *s* of a plane curve defined by *r = f(φ)*, in polar coordinates, is implicitely computed in Euclidean metric. The equation of motion on this curve is given elsewhere. We can also consider that the length *s* is the affine parameter of this curve. Therefore, the curve is defined by two functions: *r(s)* and *φ(s)*. As *s* is not used in the equation *r = f(φ)*, without altering the relation *r = f (φ)*, we have the freedom to apply a gauge transformation on this affine parameter, for taking into account the physical effect of the potential of Newtonian gravity. This will define another curve, in another space, sharing the relation *r = f(φ)* , with the Euclidean curve.

Emergence of a physical spacelike dynamic parameter

As the Newtonian time *t,* and the affine parameter *s* of Painlevé's formalism are both dynamic physical parameter, notwithstanding with the difference in nature of space and time, this will make ontologically possible their equivalence[8]. Unlike to Newtonian formalism where the time is universal here the dynamic parameter defined by Painlevé's formalism is, independently, dedicated to each observer, in the same way that the proper time is dedicated to each observer in relativity. The Painlevé formalism eliminates the metaphysical universal time from the classical mechanics. Moreover, as the conformal factor curving this metric is fully determined by the gravitational field, the latter transfers its physical attribute to the spacelike dynamic parameter of this phenomenology.

The special case of radial geodesic motion

[8] The timelike dynamic parameter t, measures the time Δt for traveling over the segment XY of the ellipse. The spacelike dynamic parameter s, measures the corresponding geodesic spatial length Δs, of the segment XY, in the conformal space, curved by the gravity, which according Painlevé's proposal, would be the physical space.

In this case, the duality degenerates to the relation $t = i.s$, the Newtonian time is the imaginary counterpart of the affine spacelike parameter of the Painlevé formalism. See annex for more details.

Annex 1: Detailed calculations [9]

Spatial curve of the geodesic according Painlevé's formalism

The Painlevé metric is

$$d s^2 = k\left(\frac{M}{r}+h\right)\left(d r^2+r^2 d\varphi^2\right) \cdot \qquad (A\text{-}0)$$

As the metric does not depend on φ, a Killing vector $\boldsymbol{R}$ associated to the conserved angular momentum L exists. Its components are $R^\mu = \{0, 1\}$. As $R_\mu = g_{\mu\nu} R^\nu$, this yields

$$L = R_\mu \frac{d\varphi}{d s} = k\left(\frac{M}{r}+h\right) r^2 \frac{d\varphi}{d s} \to \left(\frac{d\varphi}{d s}\right)^2 = \frac{L^2}{k^2\left(\frac{M}{r}+h\right)^2 r^4} . \qquad (A\text{-}1)$$

The relation between t and s may be deducted from the Hamiltonian in classical mechanics, as Painlevé recalled, in his article published at the "Académie des Sciences" on 05/01/1922.

$$\frac{1}{2}\left(\frac{d r^2}{d t^2}+\frac{r^2 d\varphi^2}{d t^2}\right)-\frac{M}{r}=h=c s t e \to d t^2=\frac{d r^2+r^2 d\varphi^2}{2\left(\frac{M}{r}+h\right)}=\to d t^2=\frac{d s^2}{k\left(\frac{M}{r}+h\right)2\left(\frac{M}{r}+h\right)} .$$

The relation between C and L will be given by

$$L^2 = k^2\left(\frac{M}{r}+h\right)^2 r^4 \frac{d\varphi^2}{d t^2\left[k\left(\frac{M}{r}+h\right)2\left(\frac{M}{r}+h\right)\right]} = \frac{k}{2} r^4 \frac{d\varphi^2}{d t^2} = \frac{k}{2} C^2 . \qquad (A\text{-}2)$$ [10]

.

Inserting L in the Painlevé metric divided by ds^2, yields

$$\left(\frac{d s}{d s}\right)^2 = 1 = k\left(\frac{M}{r}+h\right)\left[\left(\frac{d r}{d s}\right)^2+\frac{L^2}{k^2\left(\frac{M}{r}+h\right)^2 r^2}\right] .$$

$$\left(\frac{d r}{d s}\right)^2 = \frac{1}{k\left(\frac{M}{r}+h\right)} - \frac{L^2}{k^2\left(\frac{M}{r}+h\right)^2 r^2} = \frac{k\left(\frac{M}{r}+h\right) r^2 - L^2}{k^2\left(\frac{M}{r}+h\right)^2 r^2} . \qquad (A\text{-}3)$$

Let's compute *$(dr/d\varphi)^2$ = $(dr/ds)^2(ds/d\varphi)^2$* by using *(A-2)* et *(A-3)*

[9] Here, we use a relativistic well known (constants of motion) method for solving the problem, but we should use, as well, the relativistic Lagrangian equation, $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\lambda}\right)}\right)=\frac{\partial L}{\partial x^\mu}$ with the Lagrangian $L(x, dx/d\lambda) = \frac{1}{2} g_{\mu\nu}(x) (dx^\mu/d\lambda)(dx^\nu/d\lambda) = \frac{1}{2} (ds^2/d\lambda^2)$ for solving it.

[10] We see that, according to its definition in *(A-1)*, $L = C$ for $k = 2$. This was predictible as kinetic energy (of a unitary mass) , in classical mechanics is defined by $\frac{1}{2} \boldsymbol{v}^2$.

$$(\frac{dr}{d\varphi})^2 = r^2 \frac{k(\frac{M}{r}+h)r^2 - L^2}{L^2} \cdot \qquad (A\text{-}4)$$

One can write it : $\frac{L^2}{r^2}(\frac{dr}{d\varphi})^2 - [k(\frac{M}{r}+h)r^2 - L^2] = 0 \cdot$

Rearranging and dividing by kr^2 will yield :

$$\frac{L^2}{kr^4}[(\frac{dr}{d\varphi})^2 + r^2] - [(\frac{M}{r}+h)] = 0 \cdot \qquad (A\text{-}5)$$

As this equation looks like the classical equation, we will use the same method for solving it but setting

$$u = \frac{1}{r} - \frac{kM}{2L^2} \rightarrow \frac{1}{r} = u + \frac{kM}{2L^2} \cdot \qquad (A\text{-}6)$$

By using *(A-5), (A-4)* becomes :

$$\frac{L^2}{k}(u+\frac{kM}{2L^2})^4 [(\frac{dr}{du})^2(\frac{du}{d\varphi})^2 + (\frac{1}{u+\frac{kM}{2L^2}})^2] - [(M(u+\frac{kM}{2L^2})+h)] = 0 \,,$$

and with,

$$\frac{dr}{du} = -r^2 \rightarrow (\frac{dr}{du})^2 = r^4 = (u+\frac{kM}{2L^2})^{-4} \,,$$

by inserting and simplifying one get :

$$\frac{L^2}{k}[(\frac{du}{d\varphi})^2 + (u+\frac{kM}{2L^2})^2] - [M(u+\frac{kM}{2L^2})+h] = 0 \cdot$$

By developing the square in the first bracket and simplifying (the « *u.M* » terms vanish) one get :

$$(\frac{du}{d\varphi})^2 = -u^2 + \frac{k^2M^2}{4L^4} + \frac{kh}{L^2} = \alpha^2 - u^2 \quad with \quad \alpha = \sqrt{\frac{k^2M^2}{4L^4} + \frac{kh}{L^2}} \rightarrow \frac{du}{\sqrt{\alpha^2-u^2}} = \mp d\varphi \cdot \qquad (A\text{-}7)$$

The equation is easy to integrate : The solution is:

$$arccos(\frac{u}{\alpha}) = \varphi - \omega \rightarrow \frac{u}{\alpha} = cos(\varphi-\omega) \cdot$$

Coming back to coordinate *r*, will yield :

$$\frac{\frac{1}{r} - \frac{kM}{2L^2}}{\sqrt{\frac{k^2M^2}{4L^4} + \frac{kh}{L^2}}} = cos(\varphi-\omega) \rightarrow \frac{1}{r} = \sqrt{\frac{k^2M^2}{4L^4} + \frac{kh}{L^2}}\, cos(\varphi-\omega) + \frac{kM}{2L^2} \,,$$

$$\frac{1}{r} = \frac{\frac{\frac{2L^2}{kM}\sqrt{\frac{k^2M^2}{4L^4} + \frac{kh}{L^2}}\, cos(\varphi-\omega) + 1}{2L^2}}{kM} = \frac{\frac{1+\sqrt{1+\frac{4L^2h}{kM^2}}\, cos(\varphi-\omega)}{2L^2}}{kM} \,,$$

which can be written :

$$\frac{1}{r}=\frac{1+e\cos(f)}{p} \quad with \quad e=\sqrt{1+\frac{4L^2h}{kM^2}}, \quad p=\frac{2L^2}{kM}, \quad f=\varphi-\omega \cdot \qquad (A\text{-}8)$$

Relation between the spatial affine *s* parameter and the time *t* in this case.

From the relation:

$$dt^2=\frac{ds^2}{2(\frac{M}{r}+h)2(\frac{M}{r}+h)}\rightarrow dt=\frac{\pm ds}{2(\frac{M}{r}+h)}\rightarrow 2dt(\frac{M}{r}+h)=\pm ds$$

given in equation below equation (A-1) with *k =2,* by plugging the value of *r* and taking into account the definition of *e* and *p* given in equation (6), we get:

$$ds=\pm 2(\frac{M(1+e\cos(\varphi))}{p}+h)dt=\pm 2(\frac{M^2(1+e\cos(\varphi))}{L^2}+h)dt=\pm 2E(\frac{2}{e^2-1}(1+e\cos(\varphi)+1))dt$$

In the last term of the equation we set $h = E$ and used the definition of e:

$$e=\sqrt{1+\frac{2L^2E}{M^2}}\rightarrow e^2-1=\frac{2L^2E}{M^2}\rightarrow\frac{M^2}{L^2}=\frac{2E}{e^2-1}$$

For e = 0, i.e for a circular orbit we get : $ds=\pm 2\text{E.dt}$

We see that *t* and *s* are proportional in the case of a circular orbit, i.e are “equal” with adapted units.

The radial geodesic equation

The radial geodesic equation derived in the classical geometrical formalism of Painlevé

As claimed by Painlevé in equation *(A-9)*, the Newtonian gravity can be expressed by a geometric formalism similar to that of general relativity, so we will apply the method used in general relativity to derive the geodesic of:

$$ds^2 = (U+h)\,[dr^2+r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \qquad (A\text{-}9)$$

In this context, the geometric Lagrangian $L(x^\mu, dx^\mu/d\lambda)$ associated with the metric can be written

$$L(x^\mu,\frac{dx^\mu}{d\lambda})=\frac{1}{2}g_{\mu\nu}(x,y,z)[\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}]=\frac{1}{2}ds^2, \qquad (A\text{-}10)$$

where λ is the spatial affine parameter of the geodesic curve which measures the length of the curve. With the conventions of Painlevé, for the radial geodesic, the Euclidean spatial metric is:

$$g_{rr}(r)= U(r) +h.$$

This yields

$$L(r,\frac{dr}{d\lambda})=\frac{1}{2}[U(r)+h](\frac{dr}{d\lambda})^2, \qquad (A\text{-}11)$$

with the Lagrangian defined by *(A-11*), by using the Euler-Lagrange equation , we get the equation

$$[U(r)+h]\frac{d^2r}{d\lambda^2}+\frac{1}{2}(\frac{dr}{d\lambda})^2\partial_r U(r)=0,$$

either:

$$\frac{d^2r}{d\lambda^2}+\frac{1}{2}(\frac{dr}{d\lambda})^2\frac{\partial_r U(r)}{[U(r)+h]}=0\cdot \qquad (A\text{-}12)$$

This is the geodesic motion equation derived from the spatial metric *(A-9)* of Painlevé.

The radial geodesic equation derived in conventional Newtonian mechanics

In classical mechanics, the equations of motion involve derivatives with respect to time. In the definition of the Painlevé's proposal, time is absent, then how is it related to the equations of classical mechanics where the Euler-Lagrange equation can be written

$$(\frac{d}{dt})(\frac{\partial L}{\partial(\frac{dx^\mu}{dt})})=\frac{\partial L}{\partial x^\mu}, \qquad (A\text{-}13)$$

where the classical Lagrangian is equal to the kinetic energy less the potential energy, *i.e* here:

$$L(r,\frac{dr}{dt})=\frac{1}{2}(\frac{dr}{dt})^2-[U(r)+h]\cdot \qquad (A\text{-}14)$$

This shows that h is related to the total energy of the system. Applying the Lagrange equations in classical mechanics, will yield:

$$\frac{d^2r}{dt^2}+\partial_r U(r)=0\cdot \qquad (A\text{-}15)$$

Conditions to be satisfied for getting identical geodesic equations

Let's compare *(A-12)* and *(A-15)*. The form of these equations is equivalent if:

$$\frac{1}{2}\frac{(\frac{dr}{d\lambda})^2}{U(r)+h}=1\cdot$$

This can be written:

$$\frac{1}{2}(\frac{dr}{d\lambda})^2-U(r)=h\cdot \qquad (A\text{-}16)$$

Is there a relationship between the affine parameter λ, and the parameter t such as $t = a\lambda + b$, showing that the Painlevé's proposal allows to consider t as an affine parameter?

Some relativistic attributes appear in Painlevé's geometrical formalism

Let's set $t = i.\lambda.$

Plugging in equation (A-16) yields : $\frac{1}{2}(\frac{dr}{dt})^2+U(r)=-h$ ·

This is the expression of the Hamiltonian in classical mechanics giving the total energy of the test particle (of unit mass). We know that, when the potential describing the gravitational field does not depend on time, the energy is conserved on a geodesic. We have shown that whether we set $t = i\lambda$, the proposal of Painlevé is relevant. The Newtonian time t is an affine parameter of spatial curve.

Annex 2: Curvature of the conformal Euclidean space.

The space defined by Painlevè is : $d\,s^2=2\left(\frac{M}{r}+h\right)[d\,r^2+r^2\,d\,\varphi^2]$

In this equation we set, as usual because the geodesic is included in a plane, $\theta = \pi/2$. As stated before, the Weyl tensor vanishes because it is null in Euclidean space and, as it is a conformal invariant tensor, this does not change its value.

The six non vanishing Christoffel symbols values are:[11]

Γ[1, 1, 1]	$\frac{m}{-2\,m\,r-2\,h\,r^2}$
Γ[1, 2, 2]	$-\frac{r\,(m+2\,h\,r)}{2\,(m+h\,r)}$
Γ[1, 3, 3]	$-\frac{r\,(m+2\,h\,r)\,\mathrm{Sin}[\theta]^2}{2\,(m+h\,r)}$
Γ[2, 2, 1]	$\frac{m+2\,h\,r}{2\,m\,r+2\,h\,r^2}$
Γ[2, 3, 3]	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
Γ[3, 3, 1]	$\frac{m+2\,h\,r}{2\,m\,r+2\,h\,r^2}$
Γ[3, 3, 2]	$\mathrm{Cot}[\theta]$

The Riemann tensor has six different non vanishing values,

R[1, 2, 2, 1]	$\frac{h\,m\,r}{2\,(m+h\,r)^2}$
R[1, 3, 3, 1]	$\frac{h\,m\,r\,\mathrm{Sin}[\theta]^2}{2\,(m+h\,r)^2}$
R[2, 1, 2, 1]	$-\frac{h\,m}{2\,r\,(m+h\,r)^2}$
R[2, 3, 3, 2]	$-\frac{m\,(3\,m+4\,h\,r)\,\mathrm{Sin}[\theta]^2}{4\,(m+h\,r)^2}$
R[3, 1, 3, 1]	$-\frac{h\,m}{2\,r\,(m+h\,r)^2}$
R[3, 2, 3, 2]	$\frac{m\,(3\,m+4\,h\,r)}{4\,(m+h\,r)^2}$

The Ricci tensor R_{ij} and the Einstein tensor G_{ij} non vanishing different values are:

R[1, 1]	$-\frac{h\,m}{r\,(m+h\,r)^2}$
R[2, 2]	$\frac{m\,(3\,m+2\,h\,r)}{4\,(m+h\,r)^2}$
R[3, 3]	$\frac{m\,(3\,m+2\,h\,r)\,\mathrm{Sin}[\theta]^2}{4\,(m+h\,r)^2}$

[11] All these values was computed by using mathematica. Let us note that with θ = π/2, some of them may be simplified.

G[1, 1] $-\frac{m\,(3\,m+4\,h\,r)}{4\,r^2\,(m+h\,r)^2}$

G[2, 2] $\frac{h\,m\,r}{2\,(m+h\,r)^2}$

G[3, 3] $\frac{h\,m\,r\,\mathrm{Sin}[\theta]^2}{2\,(m+h\,r)^2}$

The Ricci scalar is:

$$\frac{3\,m^2}{4\,r\,(m+h\,r)^3}$$

Comments on these results.

In general relativity the Ricci tensor $R_{\mu\nu}$ and therefore the Ricci scalar R and the Einstein tensor $G_{\mu\nu}$ vanish in vacuum.

The curvature in vacuum is related by the Weyl tensor $W_{\mu\nu}$ and therefore by the Riemann tensor as the Weyl tensor is a part of it..

In the “Newtonian”solution proposed by Painlevé it is the opposite.

In vacuum, the Ricci tensor $R_{\mu\nu}$, the Einstein tensor $G_{\mu\nu}$ and the Ricci scalar R do not vanish unlike the Weyl tensor $W_{\mu\nu}$.

Therefore the Einstein equation,

$$G_{\mu\nu} = k.T_{\mu\nu}$$

where the energy-momentum tensor $T_{\mu\nu} = 0$, as we are in vacuum (no matter energy), can not be fullfilled as $G_{\mu\nu} \neq 0$! Let us recall that *k* is a postive constant ensuring the dimensional homogeneity of the equation.

This shows that, obviously, this “Newtonian” solution, whose formalism is similar to that of general relativity, is not a solution of the general relativity.

Even though they use the same geometric formalism,the two theories are different. In general relativity, the Ricci, Einstein tensors and the Ricci scalar are representing the local distribution of energy-momentum of matter or energy.

In the Painlevé “Newtonian” formalism, where it is vacuum, this would represent a fictitious matter-energy distribution, induced by the conformal factor!

Conformal factor in geometric formalism

It is also interesting to notice that applying a conformal factor in a geometric formalism is induces a curvature of this geometry.

Conclusion

This marginal contribution of Painlevé, almost unknown, included in his masterful work on general relativity, if it shows an original inspiration was treated superficially by the author.
Even though his proposal involves highly innovative ontological elements, such as the emergence of a physical dynamic parameter of geodesics, which would render obsolete the Newtonian metaphysical time and unification of the formalism for the geodesic motion, Painlevé and his contemporaries failed to explore all of the profound epistemological implications raised by this proposal. But one should not be too harsh about that, as even today, this remains widely misunderstood, being too far from our common schemes of thinking.

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