Essay on the Bell's Space ship paradox, by J. Fric[1], freely adapted from:

http://math.ucr.edu/home/baez/physics/Relativity/SR/spaceship_puzzle.html

Original by Michael Weiss 1995

http://en.wikipedia.org/wiki/Bell's_spaceship_paradox

# Bell's spaceships paradox



1 Additional personal text and comments inserted between{....} in the text or added in footnotes.

# Bell's Spaceship Paradox

## 1-Introduction

{This essay includes almost the full text of the two references listed above, I just slightly re-arranged them in order to avoid too much redundancy. I added some comments in notes and I detailed some calculations in appendixes for the curious readers. I tried to make the diagrams[2] as "exact" as possible for illustrating the arguments developed. I add some chapters for complementary analysis and remarks. I claim that the problem is part of a more general problem (extending worldlines up to make a round trip), making the current paradox related to an other well known paradox (twin paradox), both paradox relying on the relativity of simultaneity in SR.
Please note that in this essay sometimes we described a situation where there is a "string" connecting the two spaceships and we wonder whether it would break in such experiments. This may be confusing as SR tells us about space and time dilatation or contraction but to know how a physical string would break under stretch would depend of the material and geometrical parameters of this string. So, within this essay the string is just a gadget aimed to make more physical the phenomenology described. A rigorous study with a physical string is out of the scope of this essay}.

## 2- Bell's thought experiment.

John Bell described this Special Relativity paradox in the essay, "How to teach special relativity", in his collection "Speakable and Unspeakable in Quantum Mechanics." He did not originate the puzzle, but we'll call it Bell's Spaceship Paradox.

In Bell's version of the thought experiment, two spaceships, which are initially at rest in some common inertial reference frame [3] are connected by a taut string. At time zero in the common inertial frame, both spaceships start to accelerate, with a constant proper acceleration g as measured by an on-board accelerometer. Question: does the string break - i.e does the distance between the two spaceships increase?

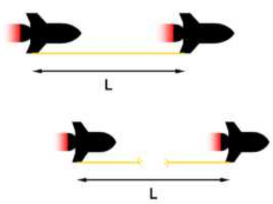


Fig 1

The situation from the viewpoint of an observer at rest: Above the spaceships at takeoff, below at 60% of the speed of light. Their distance *L* remains the same, the spaceships themselves and the string undergo a length contraction to 80% of their length at rest. Therefore the string breaks.

In a minor variant, both spaceships stop accelerating after a certain period of time previously agreed upon. The captain of each ship shuts off his engine after this time period has passed, as measured by an onboard clock. This allows before and after comparisons in suitable inertial reference frames in the sense of elementary special relativity.
According to discussions by Dewan & Beran and also Bell, in the spaceship launcher's reference system

2 Exact means "exact" according SR. I computed the equation of the curves represented and used Maxima GNU SW for graphics.
3 We will call it the "lab" frame.

the distance between the ships will remain constant while the elastic limit of the string is length contracted, so that at a certain point in time the string should break.

Objections and counter-objections have been published to the above analysis. For example, Paul Nawrocki suggests that the string should not break, [3] while Edmond Dewan defends his original analysis from these objections in a reply. [4] Bell reported that he encountered much skepticism from "a distinguished experimentalist" when he presented the paradox. To attempt to resolve the dispute, an informal and non-systematic canvas was made of the CERN theory division. According to Bell, a "clear consensus" of the CERN theory division arrived at the answer that the string would not break. Bell goes on to add "Of course, many people who get the wrong answer at first get the right answer on further reflection". [1]. Later, Matsuda and Kinoshita [5 ] reported receiving much criticism after publishing an article on their independently rediscovered version of the paradox in a Japanese journal. Matsuda and Kinshita do not cite specific papers, however, stating only that these objections were written in Japanese.

{Let's introduce first a worldline will exhibiting all the features of the problem (in fact, according to the symmetry only half is necessary for understanding these features).

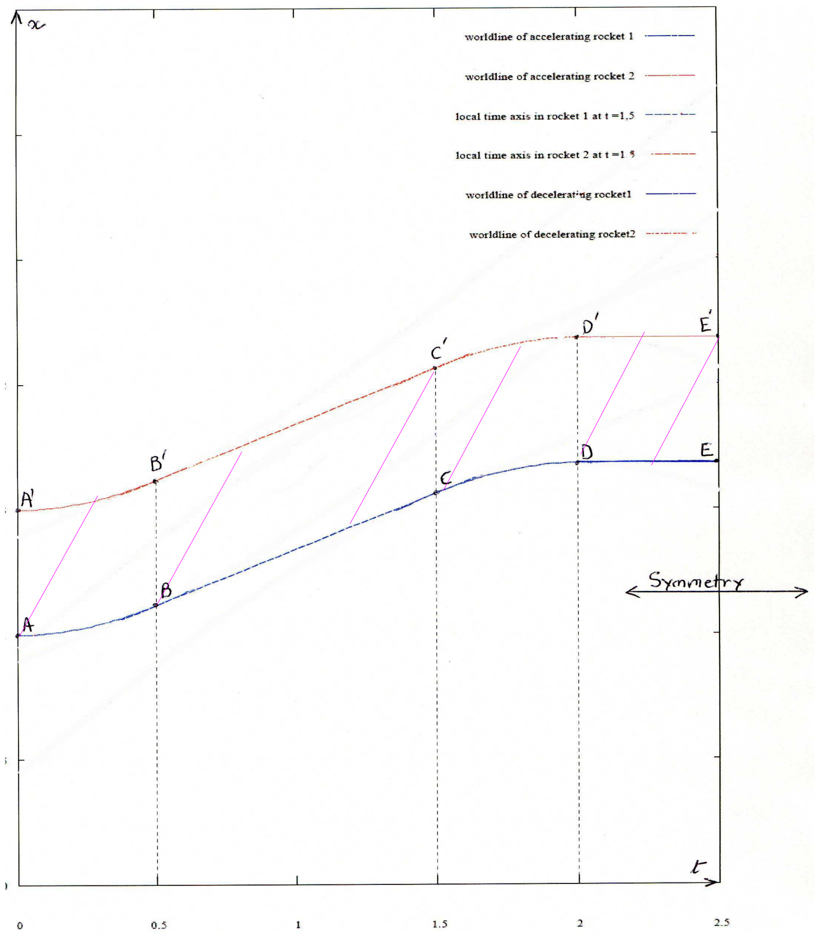


**Fig.2:** The blue line is the half closed worldline of accelerating rocket 1. On *A* , in the lab frame, the engine is fired at constant thrust moving away from the lab frame at constant accelerated speed as measured by a on board accelerometer. Then engine is stopped in *B* starting an inertial flight still moving away from the lab frame but at constant speed up to *C* where one starts to fire again the same constant thrust but in the reverse direction making the rocket to decelerate, still moving away up to *D* where the relative velocity with the lab frame vanishes and where engines are stopped again. The rocket will be in inertial flight again (D-E) remaining at constant distance of the lab frame and the rocket processes symmetrical operation to end up to a closed worldline (back at speed zero in the lab frame. At the end of such round trip the traveler in rocket 1 enjoys "proper time contraction" (he is younger than his colleague remained around the launching place in the laboratory frame). The red line is the similar worldline of rocket 2.

The set of parallel magenta skew lines are some line of simultaneity issued at the junction of the different regions corresponding to the different phenomenologies. They define transition sub-regions where the

phenomenology smoothly evolves from one to the other. We will go in more details further in the text.
What occurs on segment *A-B*, where both rocket are accelerating at constant rate[4], will be analyzed in chapter 3 (based mainly on the FAQ). We will add some comments.
Segment *B-C*, corresponding to inertial flight at constant relative velocity *v* with the lab frame, will be analyzed in chapter 4 (based mainly on the wikipedia article). We will add a study of the phenomenology at the junction *(A-B)-(B-C).*
Segment *CD*, where the rocket is decelerating is described using the same method in chapter (5).
Segment DE is relative to inertial flight in the same frame than the laboratory frame (but away from observers remained at rest).
The symmetrical part of the round trip worldline do not add any new phenomenology but is useful for exhibiting the relation with the twin paradox.}

# 3- Analysis of the basic paradox [5]

Bell asks us to consider two rocket ships, each accelerating at the same constant rate, one chasing the other. The ships start out at rest in some coordinate system (the "lab frame"). Since they have the same acceleration, their speeds should be equal at all times (relative to the lab frame) and so they should stay a constant distance apart (in the lab frame). But after a time they will acquire a large velocity, and so the distance between them should suffer Lorentz contraction. Which is it?

I think the best approach is through spacetime diagrams. I'll lay out one explanation in detail, then sketch two more. The first two are pure SR, the last has the flavor of GR but not the substance.

Don't tackle the *physical* situation directly; instead, imagine some pictures. Geometrical figures seem less prone to the "simultaneity confusion" that bedevil SR paradoxes. Once you have geometrical situation in hand, it's a piece of cake to translate it into physical terms. [Not to keep you in suspense, I'll follow each geometrical statement with the physical translation in brackets.]

If you're not familiar with spacetime diagrams, the main thing to bear in mind is how the $x$ and $t$ axes change on going from one frame to another frame in uniform motion with respect to the first. (The first frame is called the "lab frame"; the second is the frame of the "co-moving observer".)

The only real problem is drawing the $x'$-axis. It should look just like the $t'$-axis ($t'$ axis in a point $P$ is lthe tangent at $P$ to the worldline of the rocket) , slanted so that the angle between the $x$-axis and $x'$-axis equals the angle between the $t$-axis and $t'$-axis [6] . (This is assuming we choose units so $c = 1$).

The $x$-axis contains all events (spacetime points) which, according to the lab frame, occur at $t = 0$, i.e, according to the lab people, these events occur *simultaneously*. The $x'$-axis contains all events which happen at $t' = 0$, and so are simultaneous according to the moving observers. This easy graphic representation of the famous "failure of simultaneity" is one of the great strengths of Minkowski's idea.

First picture: we draw Cartesian coordinates in the plane, label the horizontal axis the $t$-axis, and the vertical axis the $x$-axis. [The $(t,x)$ system is the lab frame.] We also draw one branch of the hyperbola: $x^2 - t^2 = 1$, say the right hand branch $x = sqrt(1+t^2)$. Next we draw a parallel copy of this branch, translated upwards some fixed distance $k$, i.e. we draw $x = k + sqrt(1+t^2)$. [These two curves are the world-lines of our two rocket ships.]

For each curve, $dx/dt = 0$ at $t = 0$. [The ships are initially at rest.] Also, exercise: $d^2x/dt^2 = 1$ at $t = 0$ [7]. [Initial acceleration = 1.] Obviously $d^2x/dt^2$ is not constant. However, pick a point $P$ on one of the curves, and draw Minkowski coordinates $(t',x')$ with origin at $P$ and with $t'$-axis tangent to the curve. [The

4 Let's recall that all constant acceleration worldlines are represented by hyperbola on the Minkowski diagram..
5 Chapter 3,4,5 are are mainly a copy of the FAQ listed in the article (I just add some comments, appendixes and diagrams).
6 In other words, x'-axis is symmetrical of t'-axis around light ray issued at $P$
7 See appendix 3 for details of computation.

($t'$, $x'$) system is the frame of the inertial instantaneously co-moving observer.]

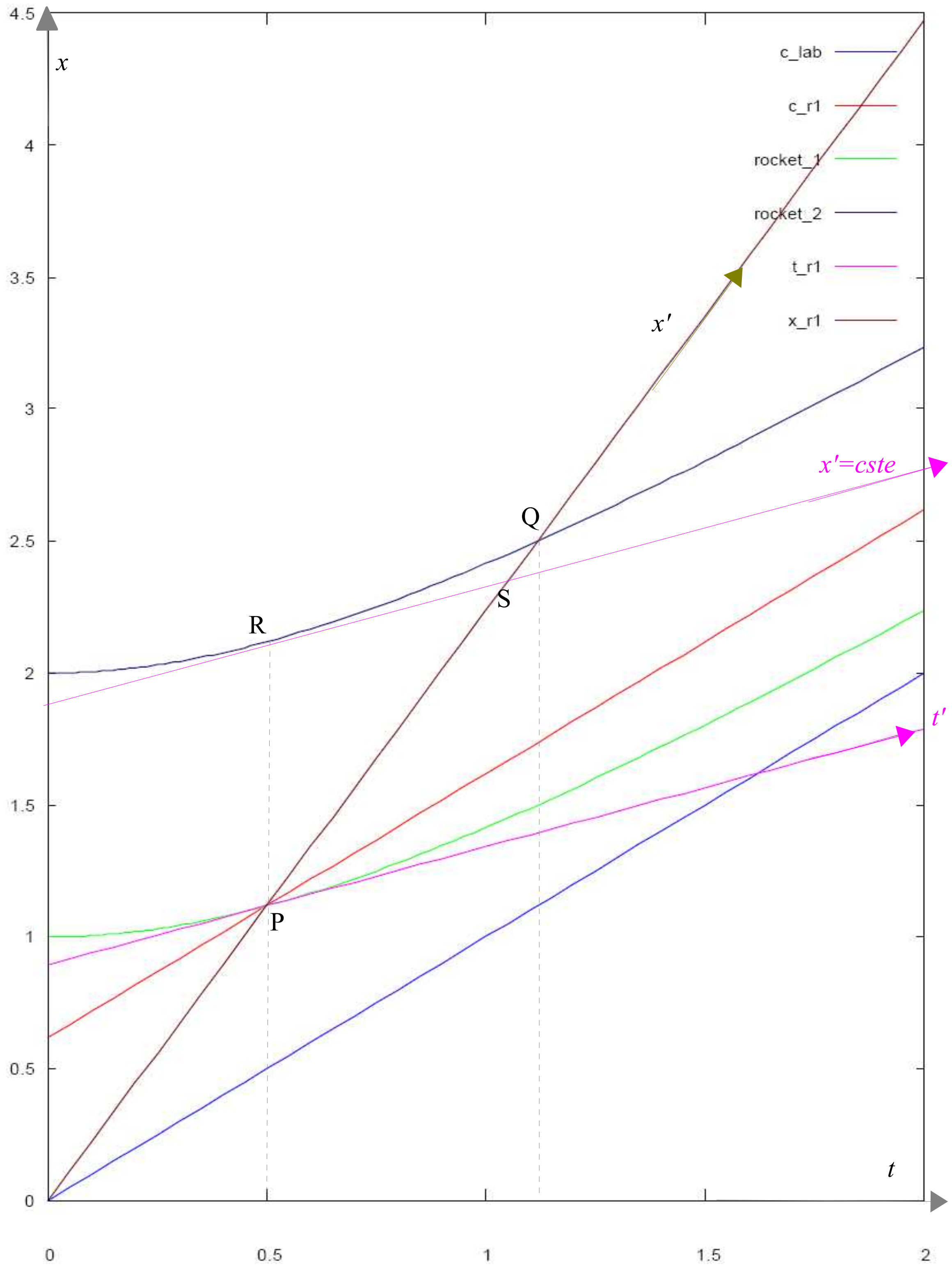


**Fig. 3**: Cartesian *(t,x)* Minkowski diagram with curves representing rocket 1 (green) and rocket 2 (blue) worldlines, rocket 1 co moving frame *t'-axis* (magenta) and *x'-axis* (brown) at *P* $(t_1,x_1)$ located on rocket 1 worldline.
A $x' = cste$ (magenta) line at $R(t_1,x_2)$, on rocket 2 (green) worldline is also represented. Be aware that the graphical length of curve AB on the diagram is not the proper time[8]: Diagram drawn using Maxima Software[9]. Here $k = 0.5$.

That is, if *P* has coordinates ($t_1$, $x_1$) in original system, then:

8 See appendix 1 for more details
9 See appendix 2 for more details

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t' =γ(t-vx) + C_0          v = dx/dt at t = t1
x' =γ(x-vt) + D_0          γ =1/sqrt(1-v2)
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where *C_0* and *D_0* are chosen so that the *P* has coordinates *(0,0)* in the primed system.

Exercise (or have a look at appendix 4): $d^2x'/dt'^2 = 1$ at P. [The acceleration of each ship is *1* as measured by inertial instantaneously co-moving observers, at all times.]

Note a few facts about this picture. First, any vertical line $t = constant$ crosses the two curves a constant *x*-distance apart; the constant is the number *k* from a few paragraphs back. [According to the lab frame, the ships stay a constant distance apart.] If we pick a point *P* on the low-hand curve, and draw Minkowski coordinates through *P* as above, then the *x′*-axis will cross the two curves at two points (*P* and *Q*) whose *x′*-coordinates differ by more than *k*. [The co-moving observer says the ships have gotten farther apart.] If *k* is small compared to γ, then *γk* is a first-order approximation to this *x′* difference. [If *k* is small enough, then the lab frame distance between the ships is approximately the co-moving distance subjected to Lorentz contraction.]

[How did the ships get farther apart, if they maintained the same constant acceleration at all times?] In the *(t',x')* coordinate system, $dx'/dt' =0$ at $t' = 0$ for the low-hand curve, but $dx'/dt' > 0$ at $t'= 0$ for the up-hand curve [10]. [The co-moving observers say the pursuing ship is momentarily at rest, but the pursued ship is moving, thanks to that old relativistic standby,failure of simultaneity. So the pursued ship is "pulling away".]

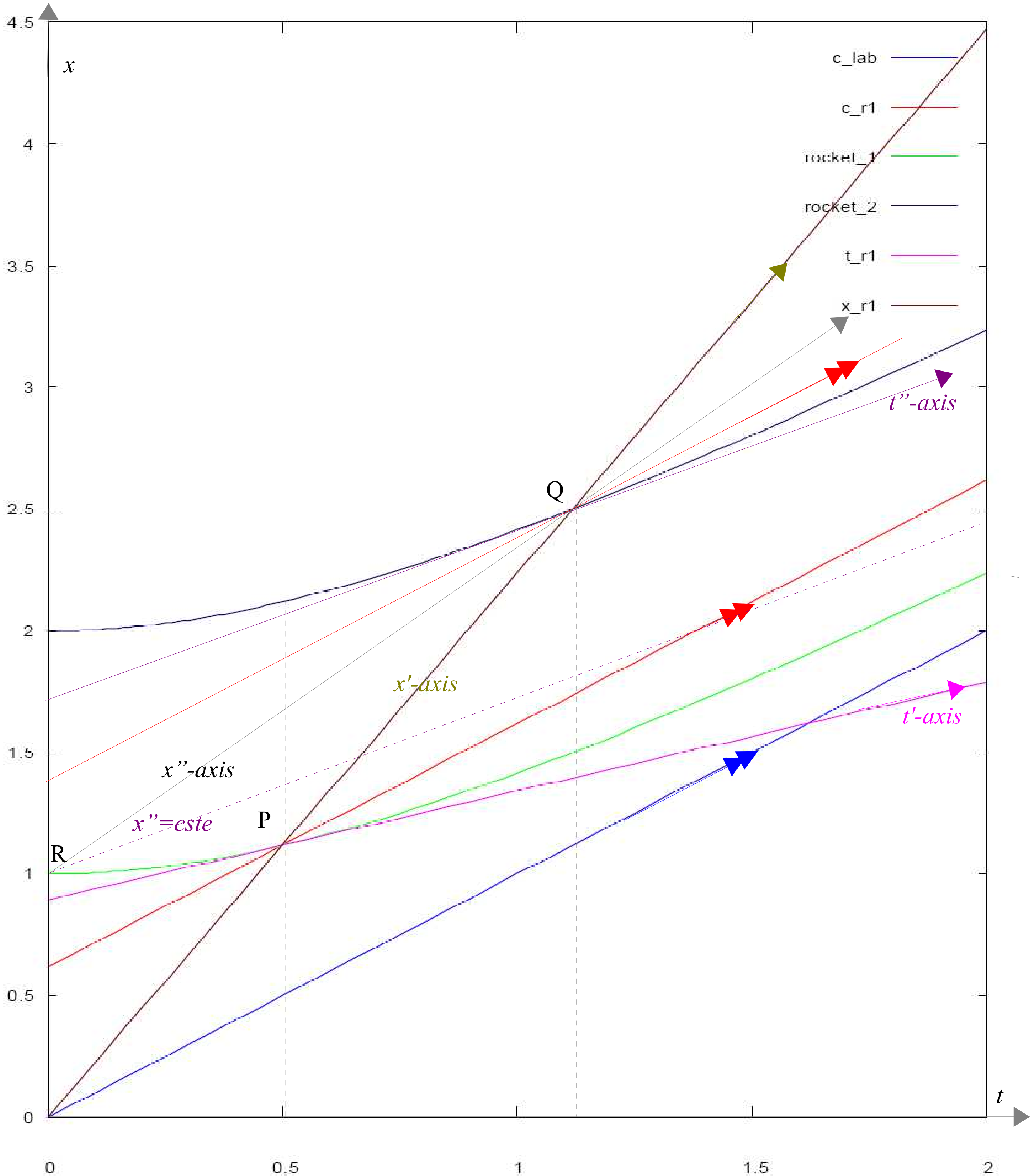
The *x′*-distance between the curves is actually slightly greater than *γk*. You can see this by a geometrical construction on fig.1. Remember that *P* is point on the low-hand (green) curve , and the *t′*-axis (magenta line) passes through *P* and is tangent to that curve. Draw a vertical line (a line $t = t_1$) through *P*; let this line cross the up-hand curve at *R*. Draw a (magenta) line parallel to the *t′*-axis through *R* (a line of the form *x′=constant*). Fact: the (brown) *x′*-axis crosses the two slanted lines, $x' = 0$ and $x' = constant$, at *x′* coordinates *0* (*P*) and *γk (S)*, respectively. Since the up-hand curve is tangent to *x′=constant* at *R*, the *x′*-axis will cross the up-hand curve at $x' > γk$. [rephrased as: According to the curvature of worldline, the curve is above the x′= constant line). Notice that the difference PQ - PS will be increasing with time! ]

{[In *t',x′ rocket 1* co-moving frame, as the *rocket 1* (green) worldline is tangent to *t′*-axis at *P*, $dx'/dt' = 0$, but at *Q, (*crossing point of *rocket 2* worldline and *x′*-axis), as the *rocket 2* (blue) worldline is not tangent to any *x′= constant* line, $dx'/dt' \neq 0$, in fact $dx'/dt' > 0$ as we can read on fig.1. In its co-moving frame, *rocket 1* (at rest by definition of co-moving frame) sees *rocket 2* ahead moving away [11], at the same time, accordingly to SR simultaneity rules ( the *x′*-axis is the line of simultaneity) ] [End of bracket convention.]

Let's have a look on what happens from the point of view of rocket 2 observer.

10 See Appendix 5.

11 Faster and faster as time increases (difference would be larger in a P' point at larger t value on rocket 1 worldline.




In rocket 2 local inertial frame ($t''$, $x''$) at $Q$ ($t''$-axis is tangent to rocket 2 worldline at $Q$ and is represented on the diagram by a brown line , $x''$-axis symmetrical of $t''$-axis around light ray worldline (double arrowed red line) at $Q$ is represented on the diagram by a black line issued at $Q$, it crosses rocket 1 worldline at R $\neq P$). So the rocket 2 observer at $Q$ sees rocket 1 observer at $R$. Let's notice the obvious break of simultaneity.

When rocket 1 is at $P$ it measures spacelike distance to rocket 2 as $PQ$. But when rocket 2 is in $Q$, it measures spacelike distance to rocket one as $QR$ which is different. We can see on the diagram that in $t'',x''$ frame (we draw a dotted magenta x"= cste line at $R$, parallel to $t''$-axis line at $Q$) that the velocity of rocket 1 is negative (green curved worldline is going under the x" = cste dotted magenta line) , rocket 2 observer at rest in $t'',x''$ frame sees rocket 1 behind her receding, which is consistent with the previous statement: they are going farther apart!).

Let's notice an interesting feature of the hyperbola defined by $x = sqrt(1+t^2)$: The x' axis (straight line of simultaneity) at any point P of the curve is a straight line passing through the origin ($x = 0, t = 0$). It is a geometrical property of hyperbola of the form x = sqrt($k^2+t^2$) as we will demonstrate later.}

This first picture interprets "two ships with the equal constant accelerations" to mean "constant for the co-moving observers, and equal in the lab frame". Note that the lab frame says that the accelerations are not constant, and the co-moving observers say the accelerations are not equal! (More precisely, any particular co-moving observer says this. The phrase "the co-moving observers" does *not* refer to a single frame of reference, unlike the phrase "the lab frame".) The lab frame says the ships maintain a constant distance from each other; the co-moving observers don't agree.

# 4- What happens after stopping engine simultaneously?

In the following analysis [12] we will treat the spaceships as point masses and only consider the length of the string. We will analyze the variant case, where both spaceships shut off their engines after some time period *T*. As we said before, in the "spaceship-launcher"'s reference system (what was called lab frame in previous chapters, here labeled *S* ) the distance *L* between the spaceships (*A* and *B* ) must remain constant "*by definition*".
{We'll draw the same kind of Minkowski diagram as in previous chapter.
Notice that we already analyze in chapter 3 the part of the diagram involving the curved worldlines of rockets (AA' -BB'). Here we'll be interested by the inertial part of the worldlines of rockets (straight lines on the diagram). As the junction of curved and straight line (transition region) should be interesting to study also, we do it at the end of this chapter.}

## 4-1 Inertial flight

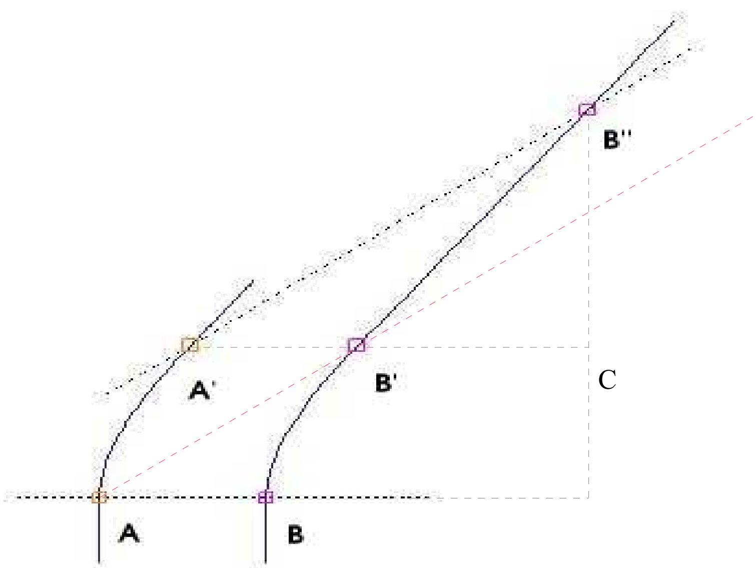



**Fig 4:** The worldlines (navy blue curves) of two observers *A* and *B* who accelerate in the same direction with the same constant magnitude acceleration. At *A'* and *B'*, the observers stop accelerating. The dotted blue lines are "lines of simultaneity" for observer *A* at *A'*. Is the spacelike line segment ***A'B"*** longer than the spacelike line segment ***AB***?
{Let's notice that the dotted red line is the line of simultaneity of observer *B* at *B',* but that the following analysis would not be relevant as we can see on fig. 4 (this line crosses the worldline of observer *A* at a

12 Based on the Wikipedia article for 6-1

point where this worldline is not globally inertial). This emphasizes the point we mentioned on fig.1. We have a phenomenology intermediate between constant acceleration and inertial when observer *B* moves from *B'* up to *B''* evolving smoothly from the first one to the latter one.
Here we will deal with the pure inertial phenomenology.}
This may be illustrated as follows. The displacement as function of time along the *X*-axis of *S* can be written as a function of time $f(t)$, for $t > 0$. The function $f(t)$ depends on engine thrust over time and is the same for both spaceships. Following this reasoning, the position coordinate of each spaceship as function of time is:

$$x_A(t) = a_0 + f(t) \qquad x_B(t) = b_0 + f(t)$$

where

- $f(0)$ is assumed to be equal to $0$,
- $x_A(t)$ is the position (x coordinate) of spaceship $A$,
- $x_B(t)$ is the position (x coordinate) of spaceship $B$,
- $a_0$ is the position of spaceship $A$ at time $0$,
- $b_0$ is the position of spaceship $B$ at time $0$.

This implies that $x_A(t) - x_B(t) = a_0 - b_0$ which is a constant, independent of time.

In other words, the distance $L$ remains the same. This argument applies to all types of synchronous motion. Thus the details of the form of $f(t)$ are not needed to carry out the analysis. Note that the form of the function $f(t)$ for constant proper acceleration is well known (see the article hyperbolic motion).

Referring to the space-time diagram (above right), we can see that both spaceships will stop accelerating at events $A'$ and $B'$, which are simultaneous in the launching frame $S$ .

We can also see from this space-time diagram that events $A'$ and $B'$ are not simultaneous in a frame co-moving with the spaceships. This is an example of the relativity of simultaneity.

From our previous argument, we can say that the length of the line segment $A'B'$ equals the length of the line segment $AB$, which is equal to the initial distance $L$ between spaceships before they started accelerating [13]. We can also say that the velocities of $A$ and $B$ in frame $S$, after the end of the acceleration phase, are equal to $v$. Finally, we can say that the proper distance between spaceships A and B after the end of the acceleration phase in a co-moving frame is equal to the Lorentz length of the line segment $A'B''$. The line $A'B''$ is defined to be a line of constant $t'$, where $t'$ is the time coordinate in the co-moving frame, a time coordinate which can be computed from the coordinates in frame $S$ via the Lorentz transform:

$$t' = (t - vx/c^2)/(1 - v^2/c^2)^{1/2}$$

Transformed into a frame co-moving with the spaceships, the line $A'B''$ is a line of constant $t'$ by definition, and represents a line between the two ships "at the same time" as simultaneity is defined in the co-moving frame.

Mathematically, in terms of the coordinates in frames $S$ and $S'$, we can represent the above statements by the following equations: $t_{B'} = t_{A'}$

results into: $x_B - x_A = x_{B'} - x_{A'} = L$ ,

$$x_{B''} - x_{B'} = v(t_{B''} - t_{B'})$$

13 In their rest frame.

$t'_{B''} = t'_{A'}$ implies: $t_{B''} - v.x_{B''}/c^2 = t_{A'} - v.x_{A'}/c^2$

In frame S', since both ends of the rope are marked simultaneously :

$\boldsymbol{A'B'} = x'_{B''} - x'_{A'}$

where:

$x'_{B''} = \gamma*(x_{B''} - v*t_{B''})$

$x'_{A'} = \gamma*(x_{A'} - v*t_{A'})$

so:

$x'_{B''} - x'_{A'} = (x_{B''} - x_{A'})/\gamma$

Calculate:

$x'_{B''} - x'_{A'} = (x'_{B''} - x'_{B'}) + (x_{B'} - x_{A'}) = (x'_{B''} - x'_{A'})*(v/c)^2 + L$ [14]

so:

$x'_{B''} - x'_{A'} = L*\gamma^2$

therefore:

$\boldsymbol{A'B''} = L*\gamma$

Thus when switching the description to the co-moving frame, the distance between the spaceships appears to increase by the relativistic factor [15]. Consequently, the string is stretched [16].

{Here the length stretch is constant (not depending on time, as in chapter 3) as we are on inertial worldlines.

Let's notice that we used Lorentz transform group for computing the stretch. Is it correct to do that?
These transforms are devoted to inertial frame in a Minkowski spacetime. In fact these infinite set of inertial frames is the Minkowski space. This spacetime is static meaning eternal, these inertial frames have been existed forever! This is what allows the symmetry of Lorentz contraction.
But here this symmetry is broken. The inertial part of the rocket frames originate from a common one, the laboratory frame. To stress this point let's consider the case where the set of infinite possible inertial frame are issue from a common one. We claim, that this is equivalent to an absolute spacetime as all the frame can be referred to the common one. So we see that we should be very cautious in using Lorentz transformation

To comfort the demonstration above we perform a direct computation of the distance in co-moving inertial frame using the basic "radar" method (exchange of light signals). This gives the same result.[17].

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14 First term of right hand equation results in geometrical considerations on the diagram $B''C/A'C = v/c$ and $B''C/B'C = c/v$.

15 ***A'B''*** is the distance between rockets measured in their rest frame (proper distance) after the end of acceleration in inertial flight as well as ***L*** was the distance between the rocket in their rest frame (proper distance) before starting moving. So we are founded to say, according to *SR* rules, that distance has increased.

16 We may wonder whether in inertial flight (no stress) a physical string with elasticity, within the elastic limit, would not recover its initial length. So forget it, we are speaking about space dilatation!

17 See appendix 7.

In fact, the above demonstration can be simplified as it is obvious that in inertial frames whether we measure the distance between rockets in the lab frame we will get *L*. But the “physical” distance between rockets should be measured in their co-moving frame where, according to Lorentz transform, we get a $\gamma$ factor yielding $l = \gamma . L$ .
As the rockets started at “physical” distance *L* in the lab frame, this stretching is physical.

And what about the length of AB segment in the rocket co_moving inertial frame? Obviously we find the Lorentz contraction as demonstrated in Appendix 7.
All of this confirm that as we are in inertial flight all the plain Lorentz transforms would apply, the “physical” stretching of space results from the fact that we know that the distance between rockets in their co-moving frame at take off was *L* }

Bell pointed out that length contraction of objects as well as the lack of length contraction between objects in frame S can be explained physically, using Maxwell's laws. The distorted intermolecular fields cause moving objects to contract or to become stressed if hindered from doing so. In contrast, no such forces act in the space between rockets.
The Bell spaceship paradox is very rarely mentioned in textbooks, but appears occasionally in special relativity notes on the internet.

An equivalent problem is more commonly mentioned in textbooks. This is the problem of Born rigid motion. Rather than ask about the separation of spaceships with the same acceleration, the problem of Born rigid motion asks "what acceleration profile is required by the second spaceship so that the distance between the spaceships remains constant in their proper frame". The accelerations of the two spaceships must in general be different [6] [7] In order for the two spaceships, initially at rest in an inertial frame, to maintain a constant proper distance, the lead spaceship must have a lower proper acceleration.[7]

# 5- Decelerating rockets.

We analyzed the problem of accelerating rocket, but what about decelerating rockets? First notice that, in our problem, when starting from rest, we may consider that rockets always accelerate (never decelerate). The direction of motion is an initial condition. But during acceleration we may stop the engine or we may reverse the thrust after the same elapsed proper time of rocket. We will switch by 180° the direction of the acceleration vector. We are tempted to say that acceleration means acceleration vector has the same direction than velocity vector and deceleration the opposite direction.
According to the general case described in fig.1 we skip inertial flight *B-C* joining segments *A-B* to *C-D* (segment *D-E* is null). This is perfectly suitable for our demonstration.

Let's consider a general diagram (fig.5) including worldlines of accelerating rockets reversing thrust after the same elapsed proper time and time in the lab frame ($t = 1$). We selected the point P in order to avoid the transient area between acceleration and deceleration, for dealing only with the deceleration problem.

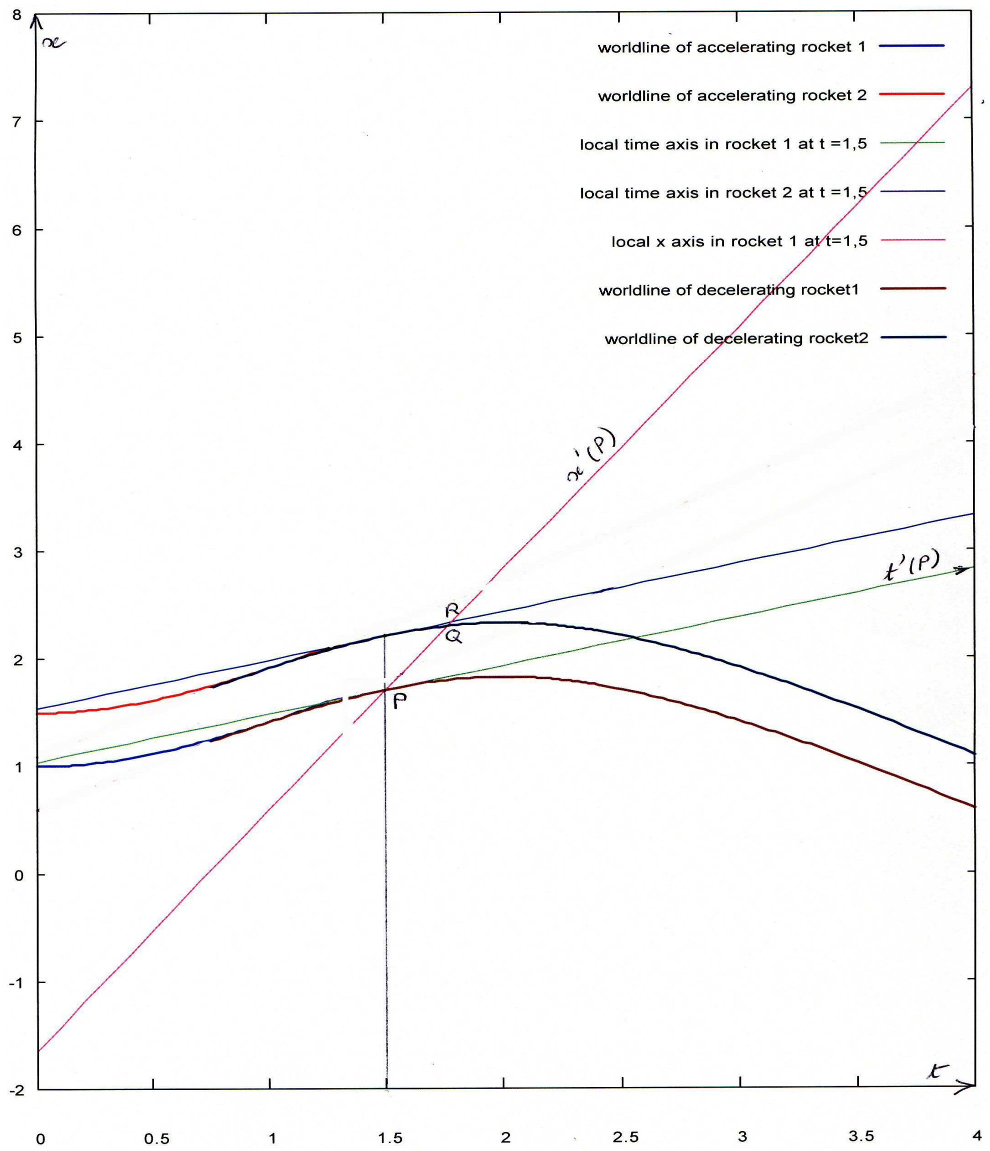



**Fig 5**: Diagram representing rocket 1 and rocket 2 accelerating in the same way than described in chapter 3 up to $t = 1$ and then decelerating at the same rate. In *P* ($t = 1{,}5$) on Rocket 1 worldline we draw *t', x'* local frame axis. We represented the same interesting points *P, Q, R* than in fig.2.

From this diagram let's make the same analysis than in chapter 3. To make things easier to see, let's make a zoom on the interesting area containing *P,Q,R* ($1.4 < t < 2$).

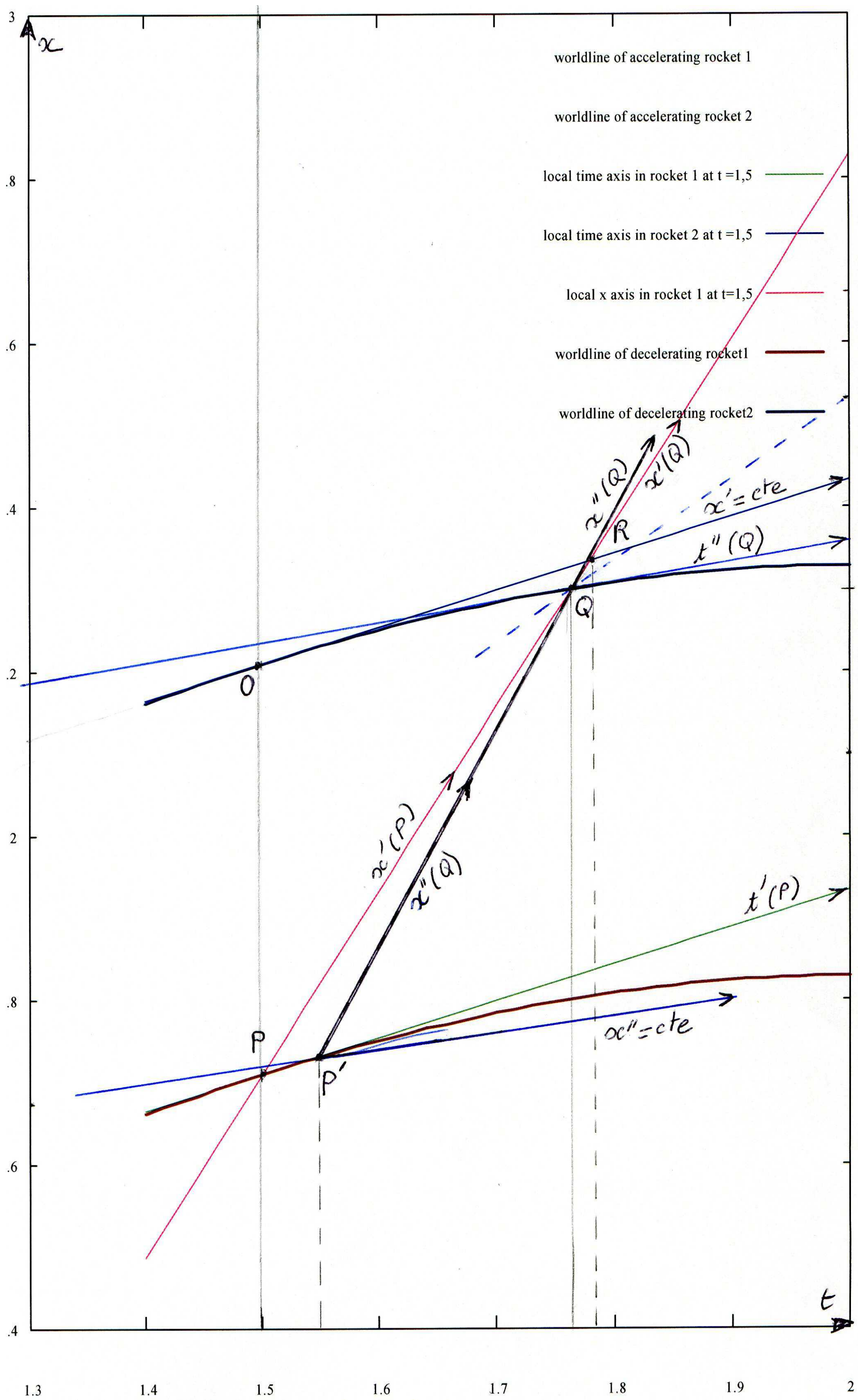



Fig.6: Zoom of the region *t =1.4, t = 2* of the previous diagram.

Note a few facts about this picture. First, any vertical line *t = constant* crosses the two curves a constant *x*-distance apart; the constant is the number *k* from a few paragraphs back. [According to the lab frame, the ships stay a constant distance apart.]

If we pick a point *P* on the low-hand curve, and draw Minkowski coordinates through *P* as above, then the *x'*-axis will cross the two curves at two points (*P* and *Q*) whose *x'*-coordinates differ by less than *k*. [The co-moving observer says the ships have gotten closer apart.] If *k* is small compared to γ, then *γk* is a first-order approximation to this $x'$ difference. [If $k$ is small enough, then the lab frame distance between the ships is approximately the co-moving distance subjected to Lorentz contraction.]

[How did the ships get closer apart, if they maintained the same constant acceleration at all times?] In the $(t',x')$ coordinate system, $dx'/dt' = 0$ at $t' = 0$ for the low-hand curve, but $dx'/dt' < 0$ at $t' = 0$ for the up-hand curve . [The co-moving observers say the pursuing ship is momentarily at rest, but the pursued ship is moving backward, thanks to that old relativistic standby,failure of simultaneity. So the pursued ship is getting closer.]

The $x'$-distance between the curves is actually slightly smaller than $\gamma k$. You can see this by a geometrical construction on fig.6. Remember that $P$ is point on the low-hand (brown) curve , and the $t'$-axis (green line) passes through $P$ and is tangent to that curve. Draw a vertical line (a line $t = t_1$) through $P$; let this line cross the up-hand curve at O. Draw a (blue) line parallel to the $t'$-axis through this point O (a line of the form $x'=constant$). Fact: the (magenta) $x'$-axis crosses the two slanted lines, $x' = 0$ and $x' = constant$, at $x'$ coordinates $0$ ($P$) and $\gamma k$ ($R$), respectively. Since the up-hand curve is tangent to $x'=constant$ at $R$, the $x'$-axis will cross the up-hand curve at $x' < \gamma k$ (according to the curvature of worldline, the curve is under the $x' = constant$ line through Q). *Notice that the difference PR - PQ will be decreasing with time!*

[In $t',x'$ *rocket 1* comoving frame, as the *rocket 1* (brown) worldline is tangent to $t'$ at $P$, $dx'/dt' = 0$, but at $Q$, (crossing point of *rocket 2* worldline and $x'$-axis), as the *rocket 2* (dark blue) worldline is not tangent to any $t'= constant$ line, $dx'/dt' \neq 0$, in fact $dx'/dt' < 0$ as we can read on fig.1. In his comoving frame, *rocket 1* (at rest by definition of comoving frame) sees *rocket 2* getting closer at the same time accordingly to SR simultaneity rules ( the $x'$-axis is the line of simultaneity) ] [End of bracket convention.]

Let's have a look on what happens from the point of view of rocket 2 observer, in rocket 2 local inertial frame ($t''$, $x''$) at $Q$ ($t''$-axis is tangent to rocket 2 worldline at $Q$ and is represented on the diagram by a blue line , $x''$-axis symmetrical of $t''$-axis around light ray worldline is represented on the diagram by a black line issued at $Q$, it crosses rocket 1 worldline at $P' \neq P$). So the rocket 2 observer at $Q$ sees rocket 1 observer at $P'$. Let's notice the obvious breakdown of simultaneity.

When rocket 1 is at $P$ it measures spacelike distance to rocket 2 as $PQ$. But when rocket 2 is in $Q$, it measures spacelike distance to rocket one as $QP'$ which is different. We can see on the diagram that in $t'',x''$ frame (we draw a blue line at $P'$ parallel to $t''$-axis line at $Q$ ) that the velocity of rocket 1 is positive, rocket 2 observer at rest in $t'',x''$ frame sees rocket 1 behind her approaching, which is consistent with the previous statement: they are going closer apart!).

We may conclude in the same way than in chapter 3 as what we described in this chapter is the anti-symmetrical phenomenon that the one described in chapter 3. So, same general conclusions would apply. This first picture interprets "two ships with the equal constant accelerations" to mean "constant for the co-moving observers, and equal in the lab frame".

Note that the lab frame says that the accelerations are not constant, and the co-moving observers say the accelerations are not equal! (More precisely, any particular co-moving observer says this. The phrase "the co-moving observers" does *not* refer to a single frame of reference, unlike the phrase "the lab frame".) The lab frame says the ships maintain a constant distance from each other; the co-moving observers don't agree.

To conclude this chapter we must say that we did not analyze the transient regions, described by the worldlines of the two rockets, on Fig.5, around t =1 corresponding to thrust reversal process.

There is no special difficulty, situation evolving smoothly from accelerating to decelerating situation passing through a neutral local point where simultaneity should be the same in the two rockets frames.

This point can be computed by equating equations of *x'_axis* for $0 < t < 1$ for rocket 1 and *x" axis* for $1 < t < 2$ for rocket 2.
Let's call *P* (on rocket 1 worldline) and *Q* (on rocket 2 worldline) these points.

Equation of x'-axis at $t_0$, $0 < t_0 < 1$, is:

$$x' = sqrt(1+t_0^2)*t/t_0$$

Equation of x" axis at $t_1$ ,$1<t_1<2$, is:

$$x'' = sqrt(1+ (t_1-2)^2)*t/ (2 - t_1 ) + b$$

The problem is symmetrical around $t = 1$, so $t_0 = 1 - t_2$, and $t_1 = 1 + t_2$
as we can check.

$$sqrt(1+(1-t_2)^2)/(1-t_2) = sqrt(1+ (t_2-1)^2)/ (1 - t_2 )$$

So for symmetry reason (around point *O* middle of vertical line between the 2 worldlines at $t = 1$) the convenient straight line should satisfy:

$$sqrt(1+t_0^2)/t_0 = sqrt(2) +0.25, (1+t_0^2)/t_0{}^2 = [ (sqrt(2) +0.25)^2] \approx 2,7696068$$
$$t_0 = 1/sqrt(\{[ (sqrt(2) +0.25)^2] -1\}) \approx 0.7517295 \approx 1- 0.248.$$

So P coordinates ($t_0$,$x$) are *(0.752, 1.28)*

*x'*-axis at *P* crosses worldline of rocket 2 at $t_1 = 1 + 0.248 = 1.248.$
Rocket 2 worldline equation for this part of the curve is:

$$x = 2*sqrt(2) +0.5 - sqrt( 1 +(t-2)^2)$$

$t_1 = 1.248$ yields $x = 2.08$

so point *Q* has coordinates ( *1.248, 2.08)*

Let's check that point is on *x'(P)* defined by $x' = sqrt(1+t_0^2)*t/t_0$

So for $t = 1.248$, $x' (P) \approx 2.08.$

*P* and *Q* are turning point in synchronization. At these point rocket 1 and 2 agree on the simultaneity of the events , so on the length of the distance between space ships. In fact each one at rest in his co-moving frame sees the other also at rest in this co-moving frame.

# 6- Round trip worldline

On the following diagram (fig.7) we see a combination of previous worldlines previously analyzed.

Here we will describe worldlines of two rockets located initially at two points of the lab frame separated by a distance *d* (which makes sense in the lab frame) simultaneously taking off (simultaneous makes sense in lab frame) and experiencing constant (in their respective co-moving frame) and identical (in lab frame) acceleration.

After some same elapsed time (as measured by their own clock in respective co-moving frame[18]) rockets stopped their engine going to inertial flight. After the same elapsed proper time in inertial flight they fire again their engine but in reversing the thrust. So the distance to lab will increase slower and slower up to a return point where it will start to decrease.

The second part of the diagram will show that we perform the same operations that in the first part in order to close the worldline (round trip).

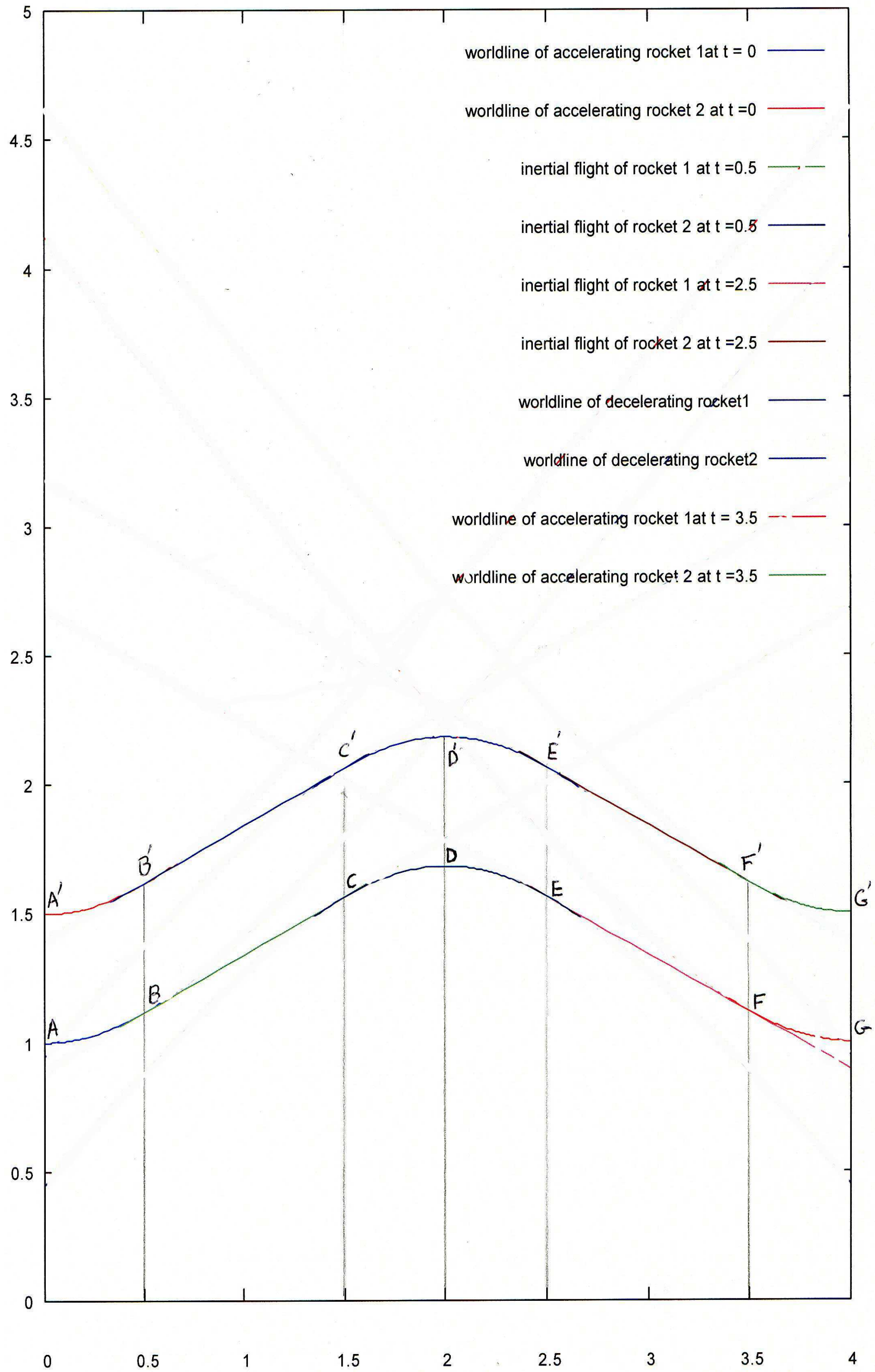



**Fig.7:** Roundtrip worldline.

18 This time is also equal in the lab frame ($t$ =0.5 in our example) as the corresponding worldlines have the same measure of affine parameter $\tau$, as we will demonstrate below.

Let's compute the "Minkowski length " of the different legs of the worldline of rocket 1.
Segment AB (length of a segment of hyperbola):

From $x = sqrt(1+t^2) \rightarrow dx = t.dt/sqrt(1+t^2)$
$ds^2 = dt^2 - dx^2 = dt^2 [1 - t^2/(1+t^2)] = dt^2( 1/1+t^2) \rightarrow ds = sqrt[1/(1+t^2)].dt$

posing $t = sinh\ \alpha) \rightarrow \alpha = argsinh\ (t)$ , $\alpha_0 = argsinh(t_0)$, $\alpha_1 = argsinh(t_1)$ and

$ds = sqrt[1/(1+sinh^2(\alpha)] = sqrt[ 1/cosh^2(\alpha)] = 1/cosh(\alpha)$.
As $dt = cosh(\alpha) .d\alpha$, $ds = [1/cosh(\alpha)]cosh(\alpha) .d\alpha, = d\alpha$.

So $s = \int_{\alpha 0}^{\alpha 1} d\alpha = \alpha_1 - \alpha_0 = argsinh(t_1)- argsinh(t_0)$
In our example we get:

$argsinh(0.5)- argsinh(0) = argsinh(0.5)= 0.481212$ as $argsinh(0)= 0$.

On the straight part (AB) of worldline the computation of s is straightforward (linear)

$ds^2 = dt^2 - v^2.dt^2$

with $v = t/sqrt\ (1+t^2) = 0.5/ sqrt[1+(0.5)^2] = 0.5/sqrt(1.25)$
$ds^2 = dt^2( 1- 0.25/1.25) = (1/1.25)dt^2 \rightarrow ds = sqrt(1/1.25).dt \rightarrow s = sqrt(1/1.25) \int_{0.5}^{1.5} dt= sqrt\ (1/1.25)$

let's now compute leg *CD*.
We see on the diagram that it is the symmetric of AB (whether we consider D at rest in the lab frame as the starting point and moving backward). The spacetime interval (s) does not depend on sign of *t*.

and all the other legs are of the same type.
So the final result comes to:
$\tau_{rocket} = 4( 0.481212) + 2\ (0.8944272) = 3.7137024$

In the laboratory frame , the elapsed time is 4.

This exhibits the twin paradox, the rockets observers are younger than theirs colleagues remained in the lab frame.
For rocket 2 we get the same result as the two function differ only by a constant which is eliminated when computing derivatives.

# 7- First variant : keep distance constant in co-moving frame.[19]

## 7-1 Accelerating worldlines.

Second picture: pick the same left-hand curve as before, but pick the right-hand curve to be [20]:

$$x = sqrt(K^2 + t^2), \quad K > 1. \qquad (7.1)$$

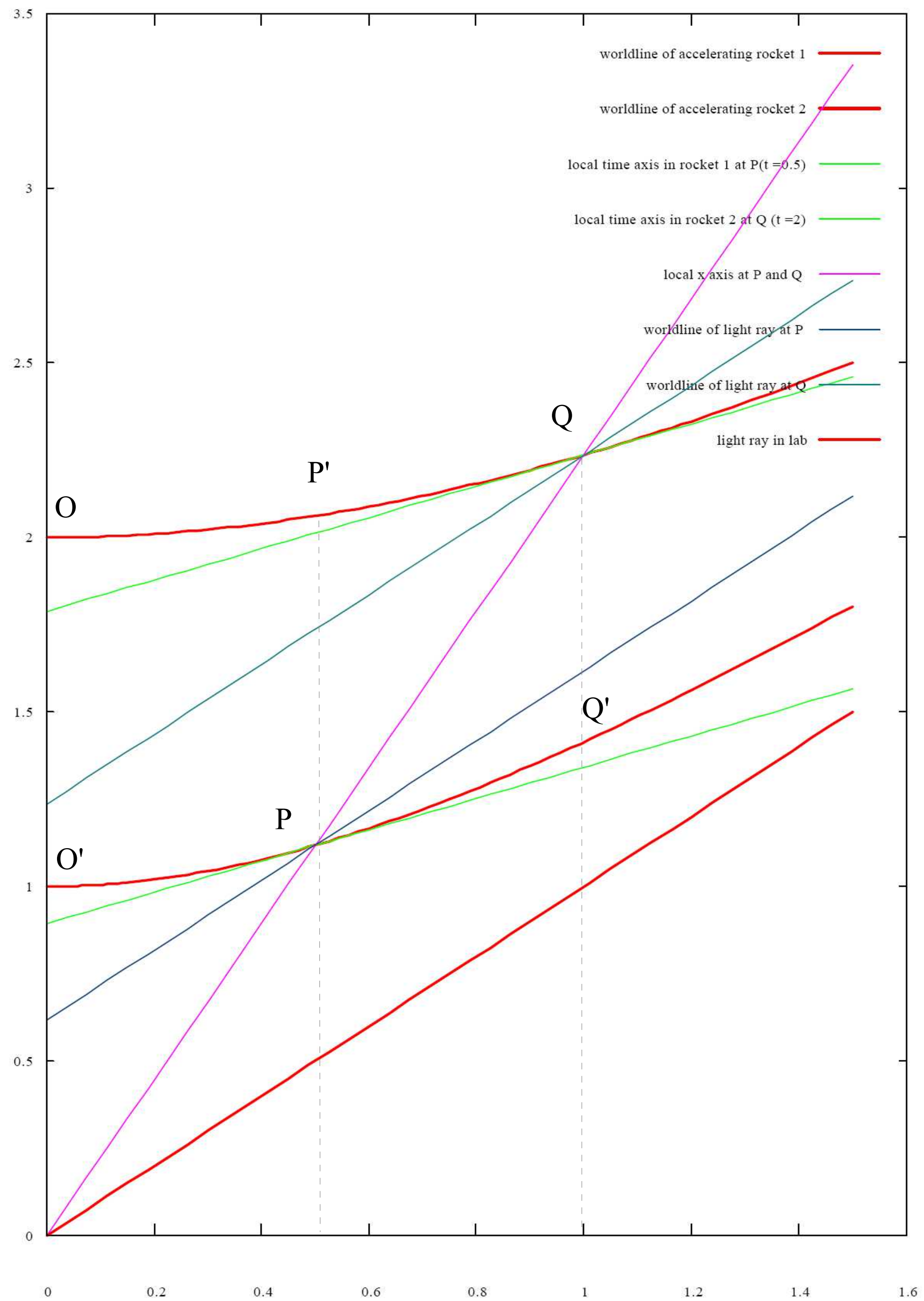



**Fig. 8**: Keeping distant constant in co-moving frame. worldlines of rocket 1 identical as before (lower hyperbola) and rocket 2 ( x = sqrt [$K^2$+$t^2$], upper hyperbola, for K =2) are represented in red thick line. The common asymptote of the two hyperbola is the (red) straight line $x = t$. At P, we represented the outgoing light ray (in navy blue) the $t'$-axis (in green) and the $x'$-axis (magenta) symmetrical of $t'$-axis around light ray.

At Q, we represented the outgoing light ray (in navy blue) the $t''$-axis (in green) and the $x''$-axis (magenta) symmetrical of $t'$-axis around light ray.

19 See FAQ originated by M. Weiss.

20 MTW discusses this case in more detail under the name "Fermi-Walker transport".

We can see that $x'$-axis and $x''$-axis are identical and that $t'$-axis at P and $t''$-axis at Q are parallel. This exhibits than there is no relative motion in both co-moving frame, (the two rockets observers agree on that): they consider the distance unchanged. But in the lab frame as we can see that QQ' < PP'< OO'. The distance suffers Lorentz contraction in the lab frame.

### 7-1-1 Here it turns out that the distance between the ships is constant according to co-moving observers.

As we said before, an hyperbola defined in Cartesian coordinates by $x = sqrt(K^2 + x^2)$ has a nice geometrical property. The $x'$-axis (line of simultaneity issued at any point P of the curve is passing through the origin ($t = 0, x = 0$). Let's demonstrate it now: Deriving (7.1) yields.

$dx/dt = t / sqrt(K^2 + t^2) = a$

The slope of the tangent at the curve is $a$.
We know that the $x'$-axis, at any $P$ on worldline defined by (7.1), is symmetrical of the tangent ($t'$-axis) around light ray worldline defined by $x = t + b$ for outgoing light ( $x = -t + c$, for ingoing light).
So the slope of the $x'$-axis at $P$ is $1/a = sqrt(K^2 + t^2)/t$.

The x'-axis general equation is then:

$x = t/a + b,$

where b is the offset when $t = 0$.

As $x'$-axis value at $P$ is $sqrt(K^2 + t^2)$, we have:

$x = sqrt(K^2 + t^2) = (sqrt(K^2 + t^2)/t)*t + b = sqrt(K^2 + t^2) + b \rightarrow b=0.$

We have demonstrated that the $x'$-axis (line of simultaneity issued at any point P of the curve is passing through the origin ($t = 0, x = 0$).

So this implies that the x'-axis at P on worldline of rocket 1 (pursuing rocket) defined by $x = sqrt(1 + t^2)$ will cross the worldline of rocket 2 (pursued rocket) at $Q$. But as at Q , the x''-axis will go through ($t = 0$ , $x = 0$), x'-axis and $x''$-axis are identical.
As at P the t'-axis is symmetrical of x' around the local light ray worldline, as well as $t''$ in $Q$ , $t'$-axis and $t''$-*axis* are parallel.
This implies that each observer, at rest by definition in his co-moving frame, sees the other also at rest. So the distance between them should not vary.

### 7-1-2 Yet the pursuer never catches his prey! (Reminds me of Achilles and the Tortoise or Keat's Grecian Urn.)

The equation (7.1) defines a family of embedded hyperbolas with the same asymptotes ($x = \pm t$ when $t \rightarrow \infty$) but different offset at $t = 0$.
These hyperbolas do not have any crossing point, as it is easy to see on the equation (7.1). The same value of $t$ yields always a different value of $x$ (for finite values).

### 7-1-3 The lab frame people measure a Lorentz-contracted distance.

As these embedded hyperbolas within the same asymptotes are not parallels in the ($t$, $x$) Cartesian diagram, it is obvious that the distance D as measured on vertical line at $t$ = constant are depending on $t$ (not equal).

### 7-1-4 The co-moving observers again say that the ships maintain constant acceleration[21].

As the computation is quite long, it is made in an appendix.

### 7-1-5 Both lab frame people and co-moving observers find that the pursuer accelerates at a greater rate than the pursued.

It is easy to see on fig. 8 than the inner hyperbola (worldline of rocket 2) has a curvature smaller than the rocket 1worldline hyperbola. The correct computation in appendix 6 shows that constant acceleration (in co-moving fame) is 1 for rocket 1 and $k^{-3/2}$ for rocket.

### 7-1-6 Explanation of this strange phenomenology

It looks quite strange that when the two rockets are accelerating in the same way in their respective co-moving frame their distance is growing and when the pursuing rocket is accelerating faster the distance may remain the same. We will discuss this point at the end but notice first that the simultaneity criterion of the SR using light signal for performing such operation would be involved.

Just let's recall how this criterion works[22].

A light signal is emitted in co-moving frame 1 at point *A* received and reflected without delay at point *B* in co-moving fame 2 and received again at *C* in co-moving frame 1. One says that the middle of worldline *AC* is synchronous of B.

But as the rockets are accelerating at constant rate, we are facing to a non linear problem.
Roughly speaking, when the signal is emitted at *E* by rocket 1 the rocket 2 is getting apart further faster an faster and when it reaches rocket 2 at *Q* and is reflected rocket 1 is going toward this light signal also faster and faster and at a different rate than for first part of the trip. So the round trip path is the sum of two parts which are different.

The middle of worldline *ER* in affine parameter (not only hyperbolic geometry, but also curved geometry as, unlike in the case of pure Lorentz transformation between two inertial frame, the ratio between affine parameter measurement and length of the curve on the diagram is not constant) in not the middle of the curve representing the worldline on the Minkowski diagram which represents the worldline in euclidean geometry.

Let see fig. 9 below for illustration.

21 See appendix 6

22 Other criterion can be used for length computation ( foliation of spacetime at constant time). See [ Gautreau-Hoffman]

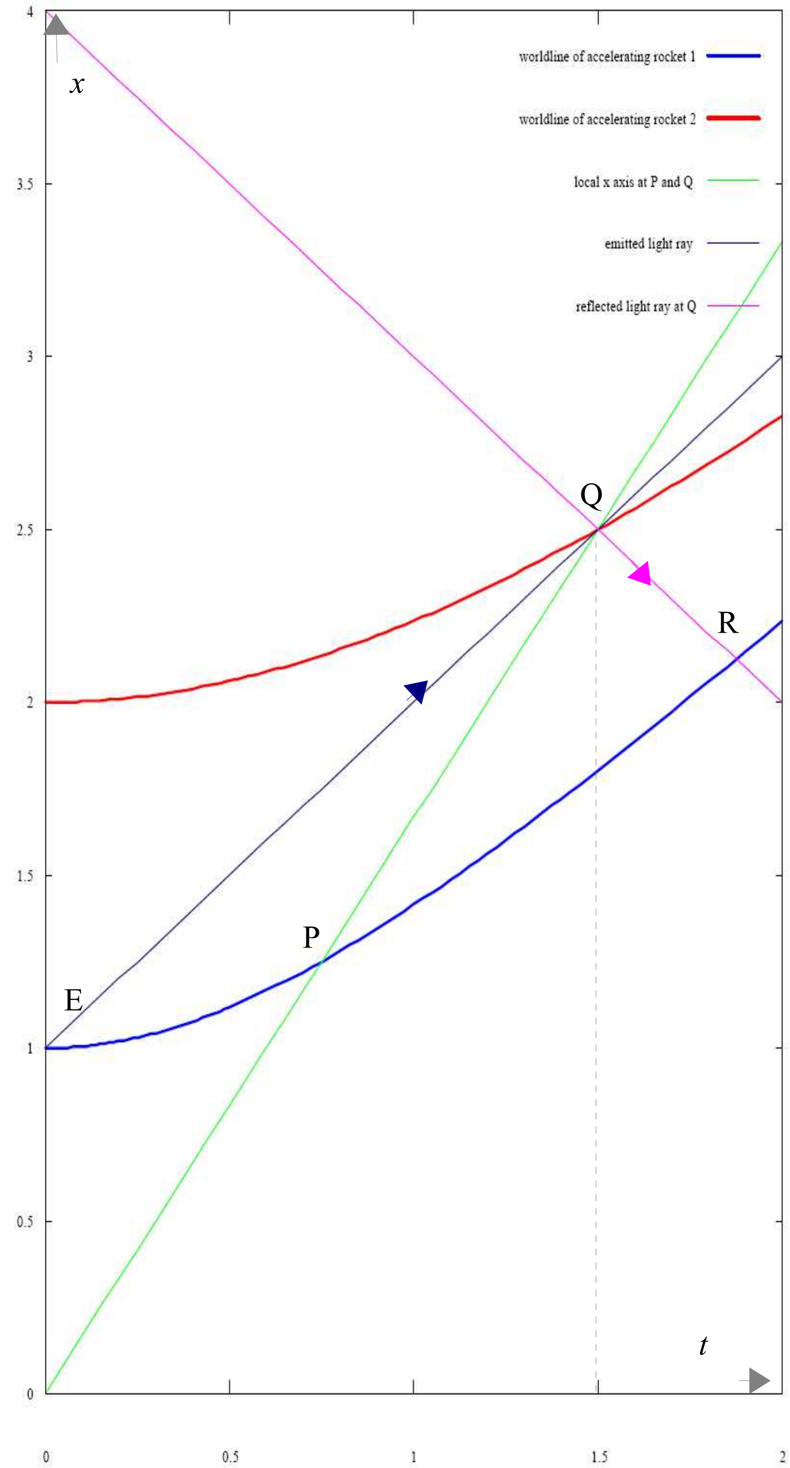



**Fig 9**: Simultaneity criterion in SR: A light ray emitted at E on rocket 1 worldline, is reflected at Q on rocket 2 worldline and received again at R on rocket 1 worldline. Point event Q is simultaneous with point event P, middle of E .
We see on the diagram that the picture does not reflect that P is the middle of ER.

In fact this solution will be introduced more formally when speaking about the Rindler space-time.

# 7-2 Round trip worldlines

Now let's wonder about what happen whether we perform a sequence of acceleration and deceleration in order to complete round trip worldlines.

In this solution the worldlines are members, defined by one parameter ($k^2$), of a family of hyperbolas having same asymptotes. This is different from the basic example where hyperbolas were just translated.

Notice also that as the rockets enjoy different acceleration, at a same time coordinate *t* in the lab frame, their co-moving frame, the elapsed time will be different. The previous scenario (the rockets reverse the thrust after an equal elapsed time since taking off) would be reconsidered.
We will have to decide when each rocket reverse the thrust, what would be the reverse thrust, and depending on the choice we will have several solutions.

We will not explore all the possibilities but we would like to see whether there exist a solution where distance in co-moving frame remain constant during the whole trip.

The geometry of the solution (embedded hyperbolas with same asymptotes) suggest that for the deceleration path the part of the rockets in the story should be reversed.

At first Rocket 1 chasing rocket 2 should have a greater proper acceleration for keeping distance constant but when decelerating this should be reversed.
Whether we like to keep the distance constant (in co-moving frame) this should be also true at the junction of the segment of the worldlines, the common x' axis, line of simultaneity which appears to be the boundary of accelerating and decelerating worldlines. All these remarks lead us to consider the following solution.

Starting from fig.8, let's call the origin *A* and extend the common *x'*-axis to the right passing through *A (0, 0)*, $P\ (t_1, x_1)$ and $Q(t_2, x_2)$. It crosses a vertical line at $A'\ [t_0 = (t_1+t_2),\ x_0 = (x_1+x_2)]$.

Point *A'* will play the part of *A* but for the decelerating worldlines (the asymptotes of the hyperbolas corresponding to decelerating worldlines will be lines issued from *A'* at 45°).

One can see the symmetry, around the point *M* middle of *PQ*, in the proposed solution.

We will demonstrate that this solution will fulfill our purpose and later we will try to find an intuitive explanation for such strange phenomenology.

Acceleration of rockets are different in both co-moving frame and lab frame but their distance remain the same!

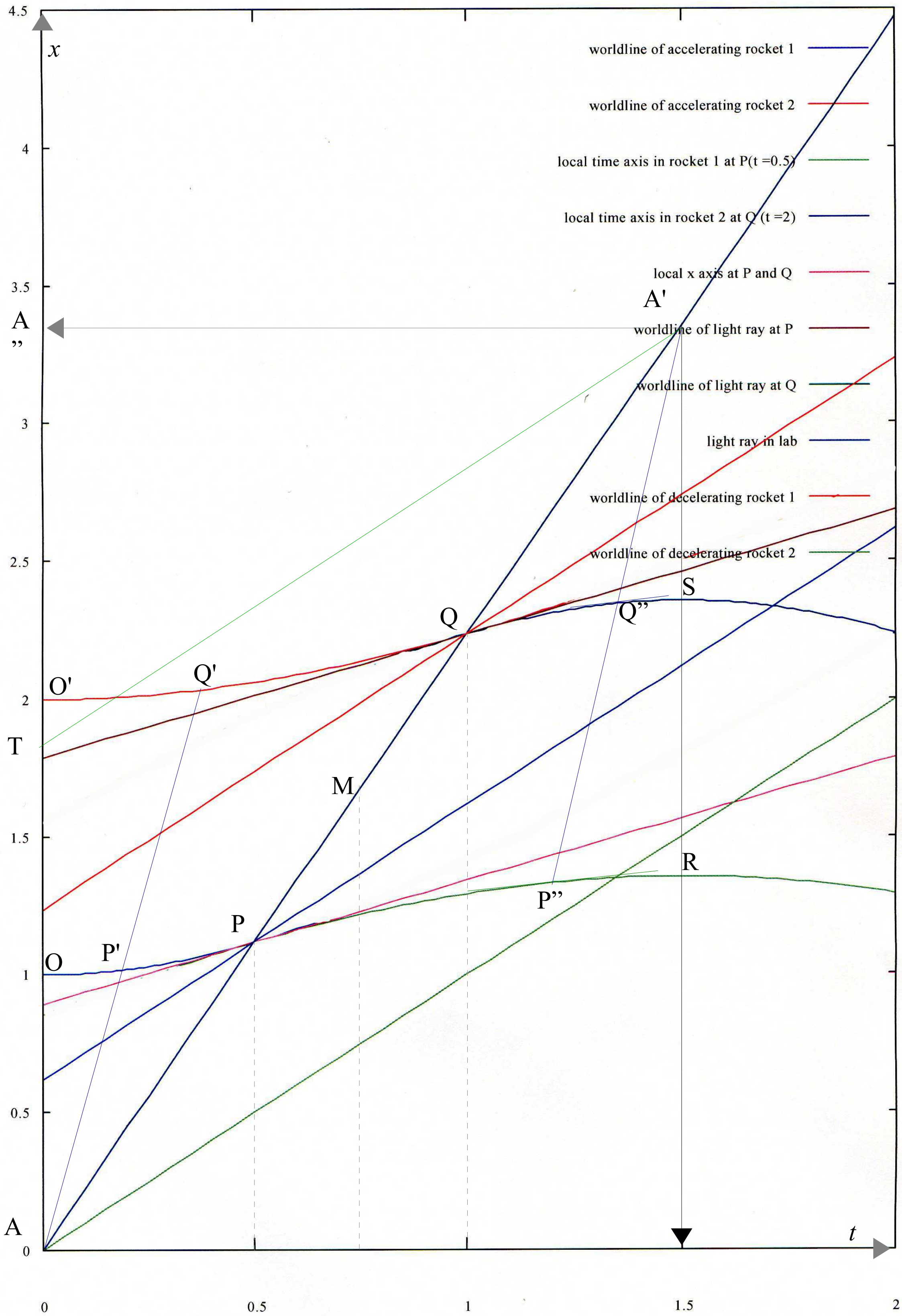



**Fig.10**: Round trip worldlines: One can see, on the first half of the worldline, that distance between rockets measured in the lab frame is decreasing, then as second half is symmetrical this distance in the lab frame would increase up to recover its initial distance.

The distance decreases in the first half as acceleration of pursuing rocket is higher than these of the pursed rocket so when accelerating distance between them will decrease and conversely.

As the acceleration of the two rockets (in their respective co-moving frame) are different, on the above diagram where they reverse the thrust at the same time (in the laboratory frame), their elapsed proper time will be different as we will demonstrate below.

But it is obvious that as velocity of rocket 1 is higher (as starting from rest at the same time, its acceleration is always higher) its proper time will be smaller when reversing thrust.

An other possible figure is to make the rockets reversing thrust after same elapsed proper time. This would give an other interesting figure, we will describe later on in this chapter.

**Proper time worldlines according fig.10 diagram:**

According to worldline computation already done in the previous chapter:
The "Minkowski length " of the different legs of the worldline of rocket 1.
Segment *OP* (length of a segment of hyperbola):

From $x = sqrt(1+t^2) \rightarrow dx = t.dt/sqrt(1+t^2)$
$ds^2 = dt^2 - dx^2 = dt^2 [1 - t^2/(1+t^2)] = dt^2( 1/1+t^2) \rightarrow ds = sqrt[1/(1+t^2)].dt$

posing $t = sinh( \alpha) \rightarrow \alpha = argsinh (t)$ , $\alpha_0 = argsinh(t_0)$, $\alpha_1 = argsinh(t_1)$ and

$ds = sqrt[1/(1+sinh^2(\alpha)] = sqrt[ 1/cosh^2(\alpha)] = 1/cosh(\alpha)$.
As $dt = cosh(\alpha) .d\alpha$, $ds = [1/cosh(\alpha)]cosh(\alpha) .d\alpha, = d\alpha$.

So $s = \int_{\alpha 0}^{\alpha 1} d\alpha = \alpha_1 - \alpha_0 = argsinh(t_1)- argsinh(t_0)$
In our example we get:

$argsinh(0.5)- argsinh(0) = argsinh(0.5)= 0.481212$ as $argsinh(0)= 0$.

For rocket 2 the same method for *O'Q* worldline gives:

From $x = sqrt(K^2+t^2) \rightarrow dx = t.dt/sqrt(K^2+t^2)$

$ds^2 = dt^2 - dx^2 = dt^2 [1 - t^2/(K^2+t^2)] = dt^2( K^2/K^2+t^2) \rightarrow ds = sqrt[1/(1+t^2/K^2)].dt$

posing $t/K = sinh(\alpha) \rightarrow \alpha = argsinh (t/K)$ , $\alpha_0 = argsinh(t_0/K)$, $\alpha_2 = argsinh(t_2/K)$ and

$ds = sqrt[1/(1+sinh^2(\alpha)].dt= sqrt[ 1/cosh^2(\alpha)]dt = dt/cosh(\alpha)$.

As $dt = K.cosh(\alpha) .d\alpha$, $ds = [1/cosh(\alpha)]K.cosh(\alpha) .d\alpha, = K. d\alpha$.

So $s = K \int_{\alpha 0}^{\alpha 2} d\alpha = \alpha_2 - \alpha_0 = K[argsinh(t_2/K)- argsinh(t_0/K)]$

In our example ($t_2 = 1, t_0 = 0, K =2$) this yields:

$$s = 2[argsinh(1/2)- argsinh(0)]= 2.argsinh(1/2) = 2.(0.481212) = 0.962424$$

We can see that this proper time is longer that the proper time of rocket 1 observer, as acceleration (so resulting velocity) is smaller by a $K^{3/2}$ factor as demonstrated in appendix 6 and it flies longer before reversing thrust.
According to the symmetry of the solution, we have *QS* = *OP* and *PR*= *O'Q*. So we have all we need for completing the computation.
The worldlines of rocket 1 and rocket 2 up to $t = 1.5$ (half of the total worldlines) is the sum of a *OP* worldline and a *O'Q* worldline, i.e in our example:

*OR = OP +PR = OP +O'Q = argsinh(1/2) + 2.argsinh(1/2) = 3.argsinh(1/2) = 1.443636*
*O'S = O'Q+QS = O'Q +OP = OR = 1.443636.*

For the full round trip it comes for traveler's proper time: $\tau = 2.\ 0.481212 = 2.887272$.
In the lab frame the corresponding elapsed proper time is $t = 3$.

We see the twin paradox effect (travelers are younger than static observers): $\tau_{rocket2} = \tau_{rocket1} < \tau_{lab}$

The scheduling of the flight is as follows:
Let's call *A1* initial acceleration of rocket 1 and *A2* initial acceleration of rocket 2.

Rocket 1 fires engine at constant acceleration *+A1* for a proper time equal to *0.481212,* then reverse thrust getting *-A2* during *1.924848* then reversing again thrust getting again *+A1* during *0.481212.*
At the end of the journey which lasted 2.887272 ( *0.481212 + 1.924848 +0.481212),* rocket 1 is again at rest in the lab frame returned at its starting point.

Rocket 2 fires engine at constant acceleration *+A2* for a proper time equal to *0.962424* then reverse thrust getting *-A1* during *0.962424* then reversing again thrust getting again *+A2* during *0.962424*.
At the end of the journey which lasted 2.887272 ( *0.962424 +0.962424+ 0.962424)* rocket 1 is at rest in the lab frame returned at its starting point.

Finally let's make a more formal computation of proper time in such solution.

Posing $t = \sinh(\alpha)$ and defining coordinates of points *A,P,Q* to be: $A\ (t_0 = 0, x_0 = 0)$, $P\ (t_1, x_1)$, $Q(t_2, x_2)$.

For *OP* worldline, $x = \mathrm{sqrt}(1+t^2)$, proper time is:

$$s = \int_{\alpha 0}^{\alpha 1} d\alpha = \alpha_1 - \alpha_0 = [\mathrm{argsinh}(t_1) - \mathrm{argsinh}(t_0)]$$

*Q* is the intersection of *x'*-axis at *P* and worldline of rocket 2.

Equation of *x'-axis at P* is: $[\mathrm{sqrt}(1+t_1^2)/t_1]t$
So $t_2$ is defined by :

$$[\mathrm{sqrt}(1+t_1^2)/t_1]t = \mathrm{sqrt}(t^2+k^2) \rightarrow [1+t_1^2)/t_1^2]t^2 = t^2+k^2 \rightarrow t^2\{ [1+t_1^2)/t_1^2]-1\} = k^2 \rightarrow t^2/t_1^2 = k^2 \rightarrow t_2 = k\ t_1$$

Plugging this in the proper time computation of proper time on worldline O'Q gives:

$$s = k\int_{\alpha 0}^{\alpha 2} d\alpha = \alpha_2 - \alpha_0 = k[\mathrm{argsinh}(t_2/k) - \mathrm{argsinh}(t_0)] = k[\mathrm{argsinh}(t_2/k)] = k[\mathrm{argsinh}(t_1)]$$

This confirm the result we got on the numerical application, up to the simultaneity common line, the proper time on worldline $x = \mathrm{sqrt}(k^2+t^2)$ is *k* time the proper time of worldline $x = \mathrm{sqrt}(1+t^2)$.

# 7-3 This spacetime is the Rindler spacetime.

In Rindler spacetime analysis , the equations of the trajectory of an uniform ( of magnitude $\alpha$) accelerated observer are defined in parametric coordinates as follows[23]:

$$t(\tau) = (1/\alpha)sinh(\alpha\tau)$$
$$x(\tau) = (1/\alpha)cosh(\alpha\tau) \qquad (7\text{-}3\text{-}1)$$

Where $\tau$ is proper time. So the definition here is different, in two ways, than the previous one which gave the space coordinate as a function of the time coordinate. First, $\tau$ is not a coordinate but is an affine parameter of the worldline, second we use a parametric definition.
We will show that even some results are the same, this is not strictly equivalent.
This definition is more physical than the previous one.

It is easy to check that this corresponds to a constant acceleration. Acceleration two-vector is given by:

$$a^\mu = D^2x^\mu/d\tau^2 = d^2x^\mu/d\tau^2 \qquad \text{(flat spacetime)}$$

The computation of the magnitude yields:

$$(a^\mu a_\mu)^{1/2} = \alpha$$

The trajectory of our accelerated observer satisfies:

$$x^2(\tau) = t^2(\tau) + \alpha^{-2}$$

We choose new coordinates $\eta, \xi$ ($-\infty < \eta, \xi < \infty$ ) such as:

$$t = (1/a)\, e^{a\xi}.sinh(a\eta) \qquad x = (1/a)e^{a\xi}cosh(a\eta) \qquad (x < |t|)$$

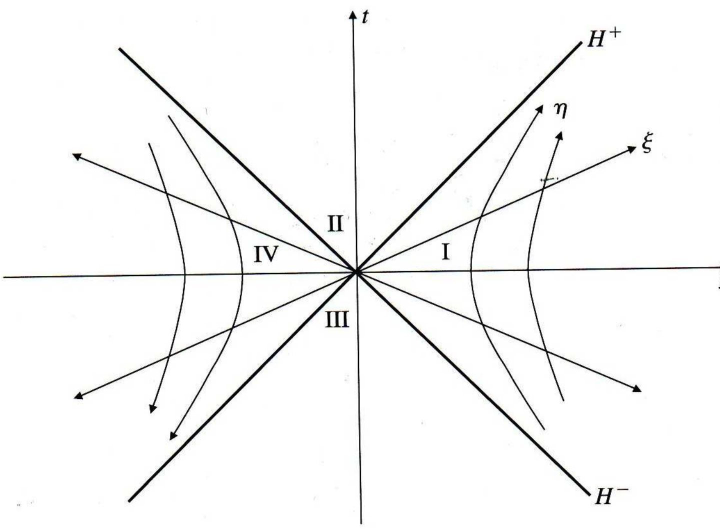



**Fig 11**: Rindler spacetime in Minkowski coordinates: Coordinates $\eta, \xi$ ($-\infty < \eta, \xi < \infty$ ) are represented on this diagram: Hyperbolas are lines of constant $\xi$ ,( for an observer with $\alpha = a$, $\eta$ is the proper time on these hyperbolas, $\xi$ is the parameter of the infinite family), straight lines issued from center of symmetry are lines of constant $\eta$.

In these coordinates the constant acceleration path, defined in (7-3-1), is described by equations:

23 For all the details, see,for instance, spacetime and geometry, p 403-406 S. Carroll, Addison Wisley 2003.

$\eta(\tau) = \alpha\tau/a$
$\xi(\tau) = [\ln(\alpha/a)]/a$

So the proper time is proportional to $\eta$ and the spatial coordinate $\xi$ is constant .
For an observer with $\alpha = a$, $\eta = \tau$ , $\xi = 0$

In these coordinates the metric is:

$$ds^2 = e^{2a\xi} ( -d\eta^2 + d\xi^2)$$

In these coordinates we can see that:

$$x^2 - t^2 = (1/a^2)\, e^{2a\xi}.\cosh^2(a\eta) - (1/a^2) e^{2a\xi} \sinh^2(a\eta) = (1/a^2)\, e^{2a\xi}$$

with $(1/a^2)\, e^{2a\xi} = K^2$ , (assuming $\xi = constant$ as well as $a = constant$), we see that this equation is the same than eq.(7.1). From $t = (1/a)e^{a\xi}.\sinh(a\eta)$, $x = (1/a)e^{a\xi}\cosh(a\eta)$,we see that $x/t$ are lines of constant $\eta$.

From $\eta = cste$ we get the spatial distance between two worldlines by integrating:

$ds = e^{a\,\xi}\, d\xi$, from $\xi = b_1$ to $\xi = b_2$ for $a = a_1$ and $a = a_2$.

We see that as the metric depends on ξ., the space coordinate is not “flat” but curved. This explain why we were in trouble with worldlines of identical constant acceleration $a$

But we are not obliged to stick to constant identical acceleration for all worldlines. The acceleration parameter is a degree of freedom. Let's select $a$ such as $a.\xi = cste = k$, and let's draw the associated family of hyperbolas in $\eta$, $\xi$ coordinates.
In this case the space coordinate is no longer curved and the computation of the distance is straightforward.
With these adapted coordinates the spatial length (at $\eta = cste$) between two of these hyperbolas of acceleration $a = a_1$ and $a = a_2$.

$$s = l = \int_{a1}^{a2} e^k\, d\xi = e^k\, [\xi_{(a2)} - \xi_{(a1)}]$$

Note that this distance does not depend of $\eta$. So when $\eta$ is varying from -∞ to +∞, generating the hyperbola, we would find always the same distance between the two hyperbolas. We have demonstrated that with such foliation of spacetime we keep the distance constant between to of the considered accelerated observers. In fact such foliation would exactly compensate the curvature of the spatial coordinate.

*As seen in this chapter, the spacetime defined in chapter 6 is definitely the Rindler (or at least one region) of the Rindler spacetime where a is the acceleration parameter and ξ is a space parameter defining one hyperbola among the infinite family of hyperbolas.*

## 7-4 Relation with Milne universe

The Milne cosmology is a pure SR theory[24], however it is also the solution of the Friedmann equation using Robertson Walker (RW) metric when density of matter $\rho$ vanishes. In this case, Friedmann equation becomes:

$$H(t)^2 = -\kappa/a(t)^2$$

where $\kappa$ is the space curvature and a the scale factor. We see that the space curvature should be negative and as H = (da/dt)/a , the equation can be simplified to :

$$da/dt = (-\kappa)^{1/2}$$

which solution is:

$$a(t) = (-\kappa)^{1/2}.t + b$$

with a(t) =0 at t =0 we get:

$$a(t) = (-\kappa)^{1/2}.t$$

the expansion law is linear. Plugging it into RW metric

$$ds^2 = -dt^2 + a(t)^2[dr^2/(1+r^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

shows that the metric of the the Milne universe can be written as follows:

$$ds^2 = -dt^2 + t^2[dr^2/(1+r^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

The 2D version is:

$$ds^2 = -dt^2 + t^2.dr^2/(1+r^2)$$

Let's perform the same coordinate transformation than the one we did for the Rindler spacetime.

$t = \cosh(\alpha)$, $r = \sinh(\alpha)$ , *we get:* $dt = \sinh(\alpha)d\alpha$, $dr = \cosh(\alpha)d\alpha$ .

$ds^2 = -dt^2 + t^2.dr^2/(1+r^2) = -\sinh^2(\alpha).d\alpha^2 + [\cosh^2(\alpha).\cosh^2(\alpha)/(1 + \sinh^2(\alpha)].d\alpha^2.$

*As* $\cosh^2(\alpha) - \sinh^2(\alpha) = 1 \rightarrow 1 + \sinh^2(\alpha) = \cosh^2(\alpha),$

$ds^2 = -dt^2 + t^2.dr^2/(1+r^2) = d\alpha^2 [\cosh^2(\alpha) - \sinh^2(\alpha)] = d\alpha^2.$

We know that $d\tau^2 = -ds^2$ so posing $\alpha = i.\tau$ we get $d\alpha = i.d\tau$, $d\alpha^2 = -d\tau^2$, the previous equation becomes:

$$ds^2 = -d\tau^2$$

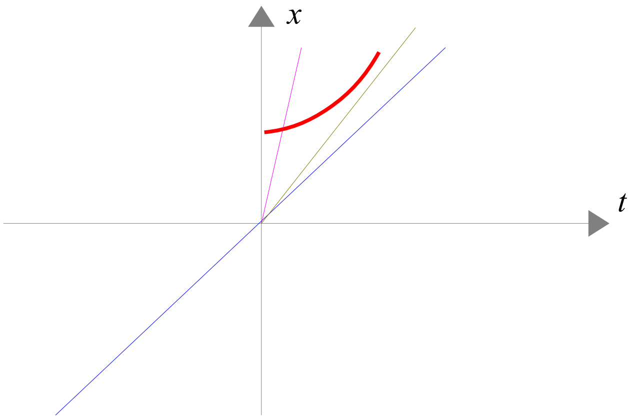
We see that the Milne spacetime is related to the Rindler spacetime by metric signature inversion.

**Complements on Milne universe.**[25]

Roughly speaking the Milne universe is a pure SR cosmological solution where the whole universe of galaxies gets created at a single point in flat spacetime. They all shoot out at different speeds.
The galaxies are treated as non gravitational test particles (no gravity). Unlike *FLRW* GR cosmologies, this universe has an outside preexisting to the explosion.

24 E.A. Milne, 1933, Z. Astrophys.6,1
25 See http://world.std.com/~mmcirvin/milne.html , for a nice description of the Milne cosmology.

To make a comparison with standard cosmologies it would be interesting to define a cosmic time in the Milne universe i.e to consider hypersurfaces of constant proper time since creation.
The equation in 2D is obvious, we just have to use the Lorentz transformation (with $c = 1$).
We sketch below the relevant Minkowski diagram.




The blue line is a light ray. The magenta line is the worldline of a galaxy of celerity $v_1$, brown line of a galaxy of celerity $v_2$. . We sketched in red a segment of constant proper time line (hyperbola)

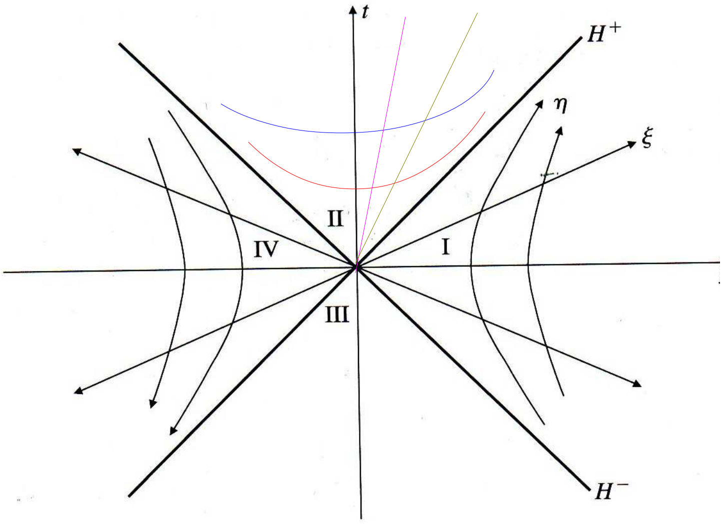
As $\tau = t/sqrt(1-v^2)$ and $v = x/t$
we get:

$$\tau^2 = t^2/(1-x^2/t^2) = 1/t^2-x^2.$$

So equation of constant $\tau$ would be the well known hyperbolas defined by:

$$t^2-x^2 = 1/\tau^2 = cste$$

*But let's notice that, unlike the Rindler spacetime, hyperbolas are not lines of constant space coordinate but instead lines of constant time.*




We drew the previous Rindler spacetime where hyperbolas in the right and left quarter are lines of constant space coordinate. In the upper ( and down not represented) quarter, hyperbolas are lines of constant proper time and magenta and brown straight lines are line of constant celerity which would play the part of spatial coordinate in this foliation of spacetime.
This confirms the relation between Rindler spacetime and Milne spacetime in this type of coordinates, time and space should be exchanged. No surprise as both are just different foliations of the Minkowski spacetime.

*This means that we are no longer facing to a timelike acceleration but to a "spacelike" acceleration.*

## 7-5 Is this distance physical?

Here we are not in the standard SR situation as some worldlines are not inertial everywhere and the lab frame is playing a central part, as rocket's worldlines originate from the lab frame. So the stretching (or non stretching, in the variant with different accelerations) of the distance should be considered physical even though it is not so easy to measure it by some rigid body.[26]

An interesting topic is that between two “truly inertial” frame (inertial for eternity) the Lorentz contraction is symmetrical (so considered as unphysical)[27] as we are not able to decide which one is originated from the other, if any (the inertial frames may have always physically existed or more realistically they may have been created at a place in spacetime which was not causally connected at Universe time of creation to what was at that time our current place: no causal connection with us: The information is beyond an event horizon) .

In both cases it is undecidable.

But in the experiences we described, we know which one has been common and as the symmetry is broken, it is sensible to say that the contraction is physical.

The interesting feature is that even in case we can not decide the SR gives the right value “of contraction”. This is the sign of a consistency of physical laws as otherwise depending on the way this situation is obtained we would have different phenomenologies.

Note that the consistency the common origin not causally connected to us,can be explicated, as even we do not know whether A comes from B or B comes from A, in both case according the invariance of physical law we get the same conclusion.

## 7-6 How physical are accelerating observers in Special Relativity?

We may have some doubt in SR as acceleration imply a physical process for doing it;
Such process should be energy and mass less as otherwise GR should apply.

26 The rigid body in non inertial frames would involve an infinite celerity for perturbation propagation (when you apply an acceleration at some place the induced perturbation should be instantaneously propagated in the rigid body. We know that such perturbation would propagate at the speed of sound in the body. This why we would rather speak of the space distance between “point-rocket” which is more appropriate in the SR context.

27 In fact it is undecidable as we are missing some information (this information mat be beyond an horizon of events

# 8- Second variant: A flavor of GR

Last picture, edging just a bit towards GR: impose the following (indefinite) metric on the (*t, x*) plane:

$$d\tau^2 = e^{2x}dt^2 - dx^2 \qquad (8\text{-}1)$$

This spacetime possesses a "uniform gravitation field".
More precisely, time-like geodesics with the initial condition $dx/d\tau = 0$ at some point *P* satisfy:

$d^2x/d\tau^2 = -1$ at *P* [28].

The geodesic equation is:

$$d^2x/d\tau^2 + \Gamma^x_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = 0$$

With $g^{xx} = 1/g_{xx} = -1$ , $g_{tt} = e^{2x}$, the only $\Gamma^x_{\mu\nu}$ non vanishing Christoffel symbols is:

$$\Gamma^x_{tt} = ½\, g^{xx}(\partial_t g_{tx} + \partial_t g_{tx} - \partial_x g_{tt}) = ½\, g^{xx}(-\partial_x g_{tt}) = ½\,(2.e^{2x}) = e^{2x} \rightarrow d^2x/d\tau^2 = -e^{2x}(dt/d\tau)^2 \qquad (8\text{-}2)$$

From *(8-1)* $dt^2/d\tau^2 = [1 + dx^2/d\tau^2]e^{-2x}$.

Plug it into *(8-2)*:

$d^2x/d\tau^2 = -e^{2x}[1 + dx^2/d\tau^2]e^{-2x} = -1 + dx^2/d\tau^2 \rightarrow d^2x/d\tau^2 = -1$ (with $dx/d\tau = 0$).

So if a lab-frame observer (that is, (*t, x*) coordinate system) lets go of an object, she'll see it drop with acceleration *1* (provided she uses a clock that keeps local time $d\tau$, e.g. an atomic clock).

Spaceships in *this* universe keep stationary by setting their engines on constant thrust.

You might be tempted to think that this universe is equivalent to ordinary flat spacetime via a coordinate transform. Not so! Take a couple of spaceships, stationary in the lab frame. They keep their engines blasting away with constant force. (I'll refrain from the obvious Star Trek jokes).

If someone steps off a ship and starts falling, he becomes (for a moment) one of our co-moving observer. (He's in free-fall, and so inertial by definition!) He falls at the universal constant acceleration of *1*, or in his humble opinion, the *ship* is accelerating at this rate.
So if our brave new world really *is* flat spacetime in disguise, then the "intrinsic acceleration" of each ship, the acceleration as measured by a co-moving observer, is always *1*. So our two ships must trace out parallel hyperbolaas in the first picture.
But then the distance between them would increase with time, as measured by the co-moving observers. But it doesn't!

So our new universe is not flat.

You can clinch the matter by computing the curvature you should get $R = -2$ (at least I did) [29].
With $t = x^0$, $x = x^1$, using Mathematica 4 with $ds^2 = -e^{2z}dt^2 + dx^2$ we find:
For the Riemann tensor two non vanishing values for components[30]: $R[0,1,1,0] = 1$, $R[1,0,1,0] = e^{2x}$,
for the Ricci tensor two non vanishing values for components: $R[0,0] = e^{2x}$, $R[1,1] = -1$,

28 See appendix 7
29 I checked by using Mathematica 4 that R = -2 with $ds^2 = -e^{2x}\, dt^2 + dx^2$ , .
30 Riemann, Ricci and Einstein tensors are symmetrical tensors. So there not only one component with the given values.

and the Einstein Tensor vanishes: $G = 0$.
Curvature Ricci scalar is R = -2.
It is quite a strange spacetime as Ricci tensor does not vanish but Einstein tensor does!

The *4-d* variant:

$$d\tau^2 = e^{2z}dt^2 - dx^2 - dy^2 - dz^2$$

is also amusing to play with.

With $t = x^0$, $x = x^1$,$y = x^2$, $z = x^3$, using Mathematica 4 with $ds^2 = -e^{2z}dt^2 + dx^2 + dy + dz^2$ we find:

For the Riemann tensor two non vanishing values for components: R[0,3,3,0] = 1, R[3,0,3,0] = $e^{2z}$,
for the Ricci tensor two non vanishing values for components: R[0,0] =$e^{2z}$, R[3,3] = -1,
and for the Einstein Tensor two non vanishing values for components : G[1,1] =1, G[2,2] =1.
Curvature Ricci scalar is R = -2.

As we can see on the Einstein (diagonal) tensor:

diag( $G_{\mu\nu}$ )= $(0,1,1,0)$

the Einstein field equations where $\kappa$ is a dimensional constant :

$$G_{\mu\nu} = \kappa.T_{\mu\nu}$$

imply a diagonal stress-energy tensor of the form:

$$diag(T_{\mu\nu}) = (0, p,p,0)$$

with zero density but non-zero (and spatially non-isotropic) pressure (the $z$ component is null) , i.e, not physically realistic[31]: Even though it is just complying with the Weak energy condition, $\rho \geq 0$, $\rho+p \geq 0$ as the pressure is confined in a plane perpendicular to the z axis, this looks not physical (we do not know physical processes giving such phenomenology).

My guess is that the Einstein empty-space equations forbid a uniform gravitational field in the above sense. I haven't checked this, though.
Historically, Einstein had some trouble with these issues for a time.
See the discussion of Born's theory of relativistically rigid bodies in Pais' biography, "Subtle is the Lord"; you may also want to look at TheRigid Rotating Disk in Relativity.

31 We may think about 2D topological defects such as walls, but these should have a time component in stress energy tensor.

# 9-Alternative analysis using Wick rotation

## 9-1 What is a Wick rotation?

In physics, **Wick rotation**, named after Gian-Carlo Wick, is a method of finding a solution to a problem in Minkowski space from a solution to a related problem in Euclidean space, by analytic continuation.
It is motivated by the observation that the Minkowski metric

$$ds^2 = -(dt^2) + dx^2 + dy^2 + dz^2$$

and the four-dimensional Euclidean metric

$$ds^2 = dt^2 + dx^2 + dy^2 + dz^2$$

are equivalent if one permits the coordinate $t$ to take on complex values.

The Minkowski metric becomes Euclidean when $t$ is restricted to the imaginary axis, and vice versa. Taking a problem expressed in Minkowski space with coordinates $x$, $y$, $z$, $t$, and substituting $w = it$, sometimes yields a problem in real Euclidean coordinates $x$, $y$, $z$, $w$ which is easier to solve.

This solution may then, under reverse substitution, yield a solution to the original problem.

Wick rotation connects quantum mechanics to statistical mechanics in a surprising way. The Schrödinger equation and the heat equation are related by Wick rotation, for example.

However, there is a slight difference. Statistical mechanics $n$-point functions satisfy positivity whereas Wick-rotated quantum field theories satisfy reflection positivity.

It is called a *rotation* because when we represent complex numbers as a plane, the multiplication of a complex number by $i$ is equivalent to rotating the vector representing that number by an angle of $\pi/2$

When Stephen Hawking wrote about "imaginary time" in his famous book *A Brief History of Time*, he was referring to Wick rotation.

Wick rotation also relates a QFT at a finite inverse temperature $\beta$ to a statistical mechanical model over the "tube" $\mathbf{R}^3 \times S^1$ with the imaginary time coordinate $\tau$ being periodic with period $\beta$.

Note, however, that the Wick rotation cannot be viewed as a rotation on a complex vector space that is equipped with the conventional norm and metric induced by the inner product, as in this case the rotation would cancel out and have no effect at all.

## 9-2 Application to the Bell paradox.

### 9-2-1 Variant 1: x = sqrt (t²+k²) worldlines.

#### 9-2-1-1Accelerating worldlines

Let's start not by the basis version but by the alternative where worldlines are defined by a family of hyperbolas $x = sqrt(k^2+t^2)$ which corresponds to the Rindler spacetime as this example is more striking.

For performing the Wick rotation we have to replace the real coordinate *t* by an imaginary one *i.t*.

$x= sqrt(k^2 + t^2) \rightarrow x = sqrt(k^2 - t^2)$ i.e $x^2 + t^2 = k^2$ which is the equation of a circle of radius *k*.

Let's sketch[32] the accelerated part of the worldline illustrated in fig.8 after a Wick rotation.

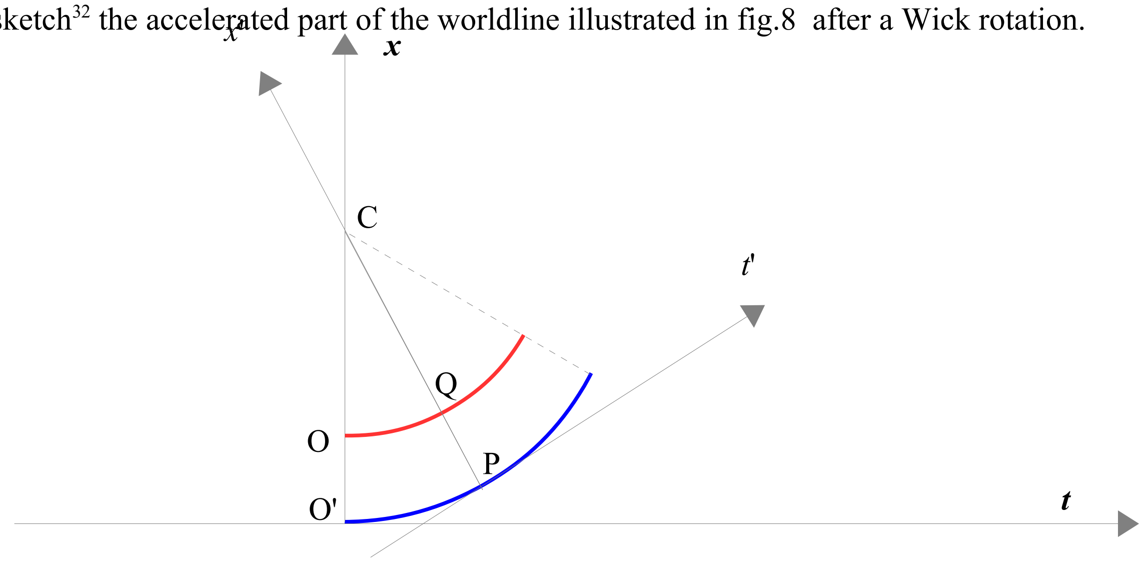



**Fig. 12**: Variant 1 accelerated worldlines represented in euclidean geometry after a Wick rotation:
Arcs of embedded hyperbolas becomes arcs of concentric circles ( center in C, radius 1 and k).
Points *O,O', P,Q* are the sames as in fig 8 (just "Wick rotated").
This is obvious as they have the same coordinates, *x,t.*
We also represented *t'-axis* and *x'-axis* of the co-moving frame of rocket 1 at *P.*
The diagram shows that the rocket 1 co-moving frame at *P (x' =0)* is also the co-moving frame of rocket 2 at *Q (x' = PQ).*

*It is remarkable that the distance PQ between the two worldlines in co-moving frame (t',x') (the shortest distance in euclidean geometry[33]) is constant. As CPQ is a common radius of the two, it is orthogonal to wordline O'P at P as well to worldline OQ at Q. This implies that the tangent to the worldlines respectively at P and Q, which are local t'-axis are orthogonal to this radius which is the local x'-axis.*

*This representation exhibits clearly this "mysterious" properties we derived previously.*
*But to make them applicable to our problem, we have to demonstrate that these properties are preserved by a Wick transformation.*

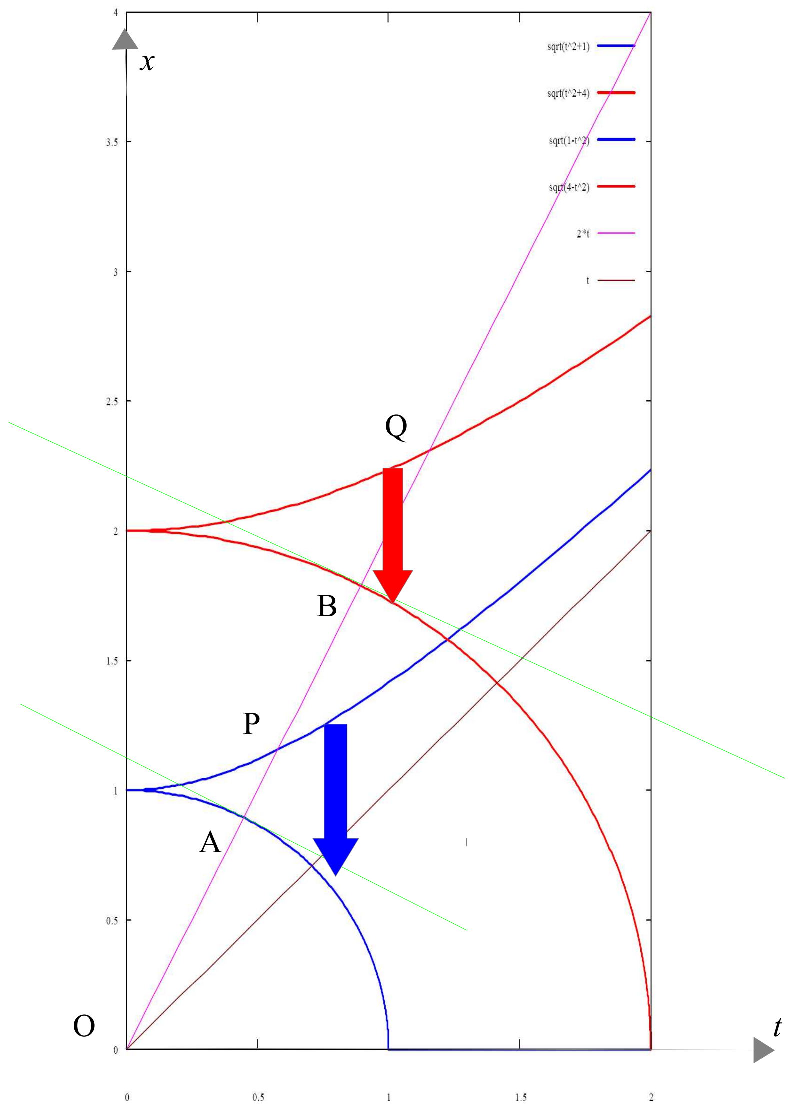
Let's recall some some important geometrical properties preserved by a Wick rotation.

- A straight line remains a straight line.

- The spatial distance conserved: this is a consequence of the definition of the two metrics and of the distance in SR (a *t = constant*): $ds^2 = dt + dx^2 = -dt^2 + dx^2$ when *t = constant (dt =0*).

- The common radius line of simultaneity issued from the center of symmetry of the circles, will remain a common line of simultaneity issued from the center of symmetry of the hyperbolas ( the tangents at intersection points which are parallel on the concentric circles remain parallel by the transformation at the corresponding intersection points on the hyperbolas).[34]

32 Fig 12,13,14 are not "exact" diagram. There have been just sketched to illustrate the result of the Wick rotation.
33 But the longest in Minkowski geometry according to the Wick rotation.
34 This is quite easy to demonstrate this property by considering in a *t,x* cartesian diagram two circles of equation $x^2+t^2 = k^2$ and $x^2 + t^2 = k'^2$ as well as the corresponding two hyperbolas of equation $x^2-t = k^2$ and $x^2-t^2 = k'^2$. A radius $x = vt$ $(v < 1)$ crosses the circles and hyperbolas on some points, we can compute, and computing the derivatives a crossing points shows that the property is satisfied.




**-Fig 12 bis**: Wick rotation, two hyperbolas $x^2 = 1+t^2$ and $x^2 = 4+t^2$ with point O as center of symmetry and common (brown) asymptote of equation $x = t$ transforms in two concentric circles of centerO (center of symmetry) of radius 1 and 2 and conversely for the inverse Wick transform. According to the symmetry , only one quarter of the figure is represented.

The magenta line is a line of simultaneity in both Minkowski and Wick rotated euclidean representation crosses hyperbolas at P and Q and Circles at A and B.

We have already demonstrated that the (green) tangents at P and Q were parallel.
The tangent at A and B are obviously parallel as they are orthogonal to a common radius of the circles. The distance between worldlines is *AB* in the Wicked rotated euclidean representation, one can see that is constant (in euclidean geometry) all along the simultaneity lines (radius of circles).

This is not obvious on *PQ* distance in the (hyperbolic) Minkowski metric as the length of such segment looks not constant on this (euclidean) diagram ($PQ \neq P'Q'$ for instance) but the fact that the tangents at P and Q are parallel ensures that this is a “perspective” effect due to the curvature of the spatial coordinate as we will demonstrate further.

### 9-2-1-2Accelerating and decelerating worldlines.

Now whether we add a decelerating path on the worldline, we see that this diagram that according to the new position of the center of the concentric circle, the exterior circle will become the inner circle and conversely. Let's sketch the worldlines of fig.10 after a Wick rotation.

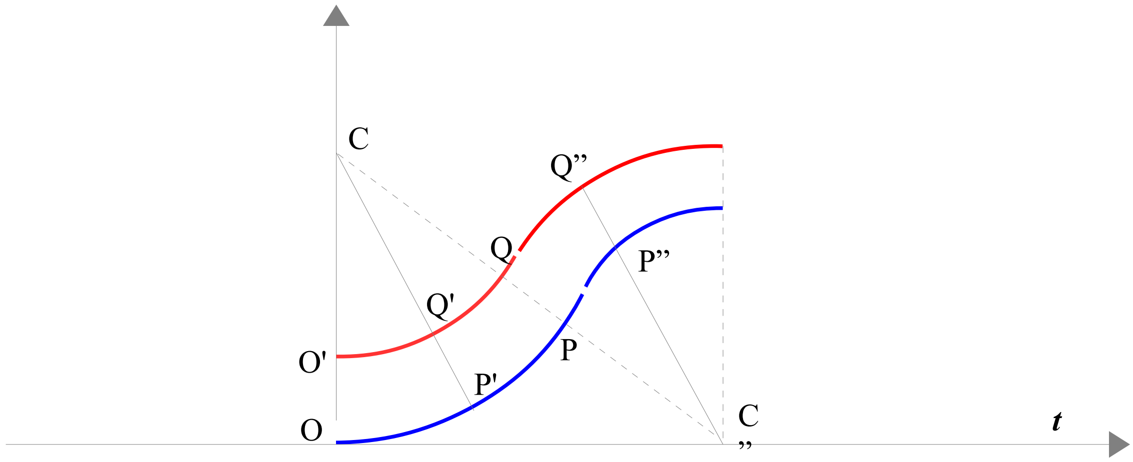



**Fig.13:** Variant 1,where we added the decelerated parts of worldlines represented in euclidean geometry after a Wick rotation: The worldlines are symmetrical around C”PQC, the common radius of accelerating and decelerating circles, boundary between acceleration and deceleration.

Arcs of embedded hyperbolas becomes arcs of concentric circles ( center in C, C”, radius 1 and k).
Points *O,O', P,Q, P',Q',P”,C”* are the sames as in fig 10 (just “Wick rotated”).
This is obvious as they have the same coordinates, *x,t.*

*Again, one can see that the distance between worldlines defined in euclidean geometry remain constant all along the worldlines and that the line of simultaneity is common along the common radius of the two circles (tangents at worllines are parallel at P' and Q', as well at P and Q as well at P”and Q” .*
*This makes obvious this mysterious property on the Minkowski diagram.*

## 9-2-2 Parallel accelerating worldlines in Minkowki diagram

Now let's go back to the basic problem, it is can be sketched as follows.

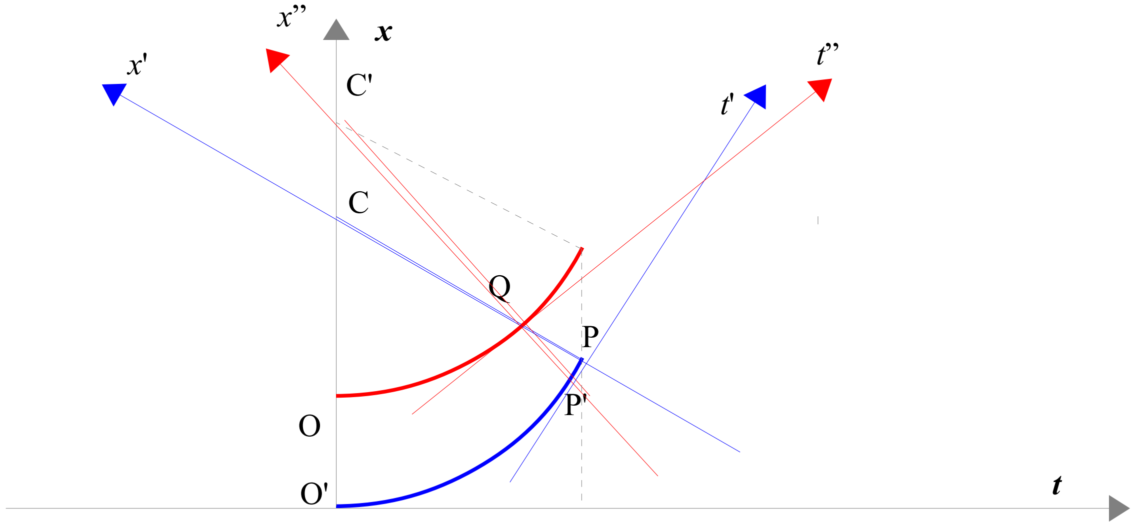



**Fig. 14**: Accelerated worldlines represented in euclidean geometry after a Wick rotation:
Arcs of translated hyperbolas becomes arcs of translated circles (centers in C, C' radius= 1 ).
We represented also *x'*-axis,*t'*-axis of rocket 1 co-moving frame at *P* as well as *x"*-axis,*t"*-axis for co-moving frame at Q of rocket 2.
There is no common radius so the distance *QP'* as seen from the worldline measured on the radius of the upper circle would be different than the distance PQ measured on the radius of the other circle.

In addition one can see that for each observer the distance between rockets is varying from the starting common value OO' (when both rockets are at rest) to respectively QP and QP' up to Q;

*Again this representation makes obvious a property which was not obvious on the Minkowski diagram.*

## 9-2-3 Parallel accelerating followed by inertial worldlines in Minkowki diagram

Now let's go back to the inertial variant of the basic problem, it is can be sketched as follows.

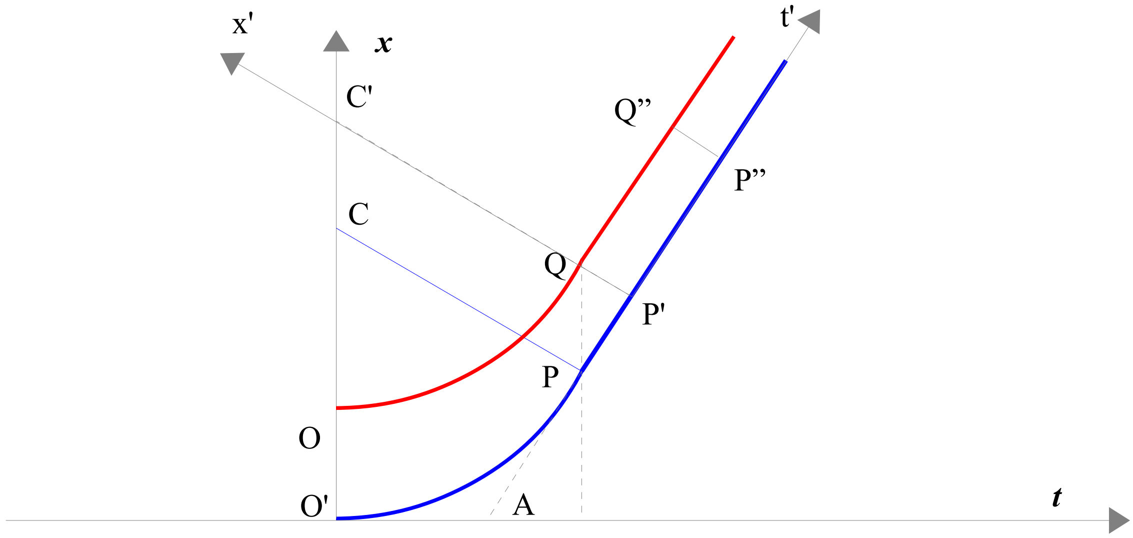



**Fig. 15**: Accelerated worldlines followed by inertial worldlines represented in euclidean geometry after a

Wick rotation: Arcs of translated hyperbolas becomes arcs of translated circles (centers in C, C' radius= 1 ), straight lines remain straight lines. The x'-axis and t'-axis of the co-moving frame is represented

In addition to comments to fig.14 for the non inertial part of the worldline, we see that after *PQ* there is a transition area up to full inertial flight starting at P'Q. After P'Q , unlike before as stated on fig.14, the distance in co-moving frame (*t',x'*) will become the same for both rockets and will be well defined.
But we see that this distance *d* is shorter[35] (on the diagram) than the initial distance *D* in lab frame (*t,x* ):

$$d = P'Q = P''Q'' < D = O'O = PQ.$$

It is easy to compute the factor of "contraction". The celerity at *P* is: $v = dx/dt = tg\ (\hat{A})$
Angles $\hat{A}$ and *PQP'* are equal, So $d = D.\cos(\hat{A})$, with $1/\cos^2 x = 1+tg^2 x$, we get:

$$1+v^2 = 1/\cos^2\hat{A}\ ,\ \cos\hat{A} = sqrt(1/1+v^2) \text{ so } d = D/sqrt(1+v^2) \qquad (1)$$

The Wick rotation from euclidean to Minkowski spacetime is defined by transforming *t* into *i.t,*

$v = dx/dt$ becomes in Minkowki metric $i.v$ and $v^2 \to -v^2$.

Plugging this in *eq.(1)* gives: $d = D/sqrt(1-v^2)$

This is the result we expected (Lorentz "contraction") showing that in co-moving frame the "proper" distance is longer than the one in the lab frame. As we started from lab frame with co-moving length equal to D, the contraction is physical. Any string binding the two rockets would be stretched!

*Again this representation makes obvious a property which was not obvious on the Minkowski diagram.*

# 10- Conclusion

Along this essay we explored some variants of the Bell's spaceship paradox. Let's summarize some important facts we encountered within this work.

**Parallel worldlines**

When the worldlines of the ship were parallel in the Minkowski spacetime (same constant proper acceleration, distance between ships remaining the same in lab frame), we noticed that, due to a lack of synchronization, according to the definition of distance in SR, it was not possible to define a "common" distance between ships at a time as the observers did not agree on a common value.
But both agree on the fact that this distance is not constant in their co-moving frame and is increasing.

Let's notice that "two ships with the equal constant accelerations" to mean "constant for the co-moving observers, and equal in the lab frame". Note that the lab frame says that the accelerations are not constant, and the co-moving observers say the accelerations are not equal! (More precisely, any particular co-moving observer says this. The phrase "the co-moving observers" does *not* refer to a single frame of reference, unlike the phrase "the lab frame".) The lab frame says the ships maintain a constant distance from each other; the co-moving observers don't agree.

Whether after some acceleration equal steps the spaceships reverse the thrust in order to decelerate, we will find the same phenomenology but this time the distance is decreasing. Whether the deceleration step is equal to the acceleration step then the distance would recover its initial value. Whether we close the loop (roundtrip) then the spaceship will recover all their initial parameters but obviously the space travelers will enjoy the twin paradox effect (elapsed proper time of both travelers equal but shorter than proper time of lab observers).

35 But according to the Wick rotation the shorter would become the longer!

We notice an antisymmetry when we switch the direction of acceleration. We will define acceleration and deceleration as follows.
When acceleration vector is pointed outwards distance increases, when pointed inwards (toward the lab frame) distance decreases.
In others words, starting from frames which are the same, acceleration makes relative absolute value of velocity to increase and deceleration make relative velocity to decrease:
Acceleration → $|v|\uparrow$ , deceleration → $|v|\downarrow$

In a second step, observers will stop their engine after some equal elapsed proper time continuing in inertial flight.
So the worldline are not totally inertial but one part of it is inertial. Sticking to the inertial part, we are able to use the SR machinery for realistic computation as this time both space ships' observers will agree upon their distance. Computing it (by different methods) in their common co-moving frame we notice that the distance is greater that the initial one. This looks physical!

Conversely, the length of an object lying in the lab frame as measured in the co-moving space-ship frame obeys to Lorentz transforms.

**Is the length stretching physical ?**

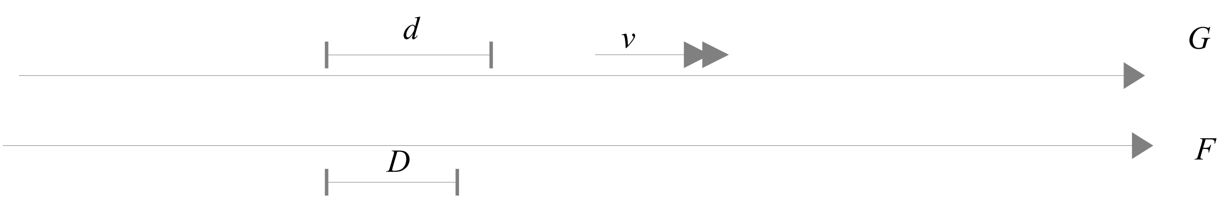
Anyway we are facing a quite interesting puzzle. We know that in pure ("eternal") SR inertial frame Lorentz contraction is symmetrical! So this is interpreted usually as a "perspective effect" as it is difficult to imagine A longer than B and B longer than A. But is it a sensible question?

How comes that such "eternal" inertial frame could exist?

Let's notice that "eternal" would mean that in our observable universe we would observe such inertial frame everywhere and at any time in spacetime. So they may have been issued from common frame, but we would never know and we would have no mean to know (for instance this event is beyond an event horizon) how and where this could have happen and what would be the original length of the object before going to this inertial frame!

A consequence of this is some undecidable principle!
.
But the physics should be consistent and even though we do know how this originates we physical laws relative to some "identical" configurations of inertial frames (same relative velocity $v$) should be the same, no matter how we got them !
And happily that's the case, as the same Lorentz transforms applies. A distance $d$ in one inertial frame measured $D$ in an other inertial suffers a Lorentz contraction by a factor $1/\gamma$. $(D = d/\gamma)$.

We do not know how two different "Lorentz" inertial frames $F$ and $G$ with a relative velocity $v$ come from but we are able to make them become one common frame by the reverse process we are talking about in this essay. Let's select the following orientation choice[36] compliant with our choice in this essay.




And positive acceleration is considered pointing to the right.

36 This choice is arbitrary but you can check that an other choice of orientation would give the same result.

Let's consider two inertial frames above. By decelerating *G* we may make it to become *F*. The length *d* in *G* frame is measured $D = d/\gamma$ in *F*. When we decelerate we have seen that the physical distance is contracted. Our computations have shown that when arriving in *F* the segment *d* would have a length contracted by a $1/\gamma$ factor i.e the length is now $D = d/\gamma$, exactly what was measured by F.

Now let's consider the other possibility: *F* accelerate in order to come to *G*. We know that physical distances would increase in this process. So *D* would be stretched by a $\gamma$ factor and its length would become $D.\gamma = d$.

By these operations we have demonstrated that no matter whether *F* comes from *G* or *G* come from *F*, as starting with two identical objects in one inertial frame whether one of these objects goes away and finally flies in an inertial frame, the physical length of both objects would obey to Lorentz transform.

This shows that, as it is well known, we have no mean for checking this "physically" without breaking he symmetry.
But in the Bell paradox the symmetry is broken, as spaceships both originate from lab frame, we have all information about the evolution of the system, it is decidable, so without the all the argument we have developed it does makes sense to say that the distance has changed.

In fact the real paradox was about the compatibility of the computation we made in the Bell's spaceship paradox which exhibited a physical stretch and the standard SR Lorentz transform where such stretching were considered as non physical and where we show that in fact this results of an undecidable principle but it should be likely physical even we have no mean to be sure except in breaking the "symmetry"[37] .

As we stated before, the consistency of the laws of physics ensures that the phenomenology of a well defined system does not depend on its history (Markovian processes).

It is quite interesting to notice that the measurement (without breaking symmetry) of an inertial moving body using light signals which seems something quite conventional predicts , via a Lorentz transform, the same result that what we get when breaking the symmetry and performing physical measurement.

The only info we do not get by the first process is how we got this system. But according to the relativity of physical laws (SR) for the inertial system it is not necessary to know it!

The fact that (no symmetry breaking) measurements using light signals accounts has the same physical consistency (taking into account Lorentz transforms) that physical (symmetry breaking) measurements exhibits the fundamental part of the light in the SR theory, which is perfectly reflected (contained) by the structure of the Minkowski spacetime.
No wonder, as SR has been built on these hypothesis (invariance of physical laws in inertial frames)!

**Time dilatation**

We have seen that time is also dilated. It is easy to understand that a clock experiencing an acceleration may being slowed down, but when after acceleration we are going to inertial frame why should a physical clock being slowed down? This looks unphysical.

The answer is in he Minkowski spacetime structure which is built on inertial frames having relative velocity and experiencing relative time and space elasticity. This structure is less trivial than it appears!
This should be explained in more details.

**GR versus SR**

37 This looks similar to QM where before an experiment the state of a system is not decidable!

GR tells us that this is matter and energy which create time and space. In other word without matter and energy do not exist.
So is it sensible to speak about time and space in SR where there is no matter?

**Is the Minkowski spacetime physical?**

According GR it would not be physical as it looks to be empty ? So should it be considered only as a limit of GR when matter energy fields are almost vanishing?

**Would antimatter save the Minkowski physical spacetime?**

An other possibility is that antimatter has a negative gravitational mass. In case there would be matter, so space and time as well as inertia and the gravitational structure of the space time should be the Minkowski space time.

This should be studied in more details.....

The loop is closed...

**Embedded hyperbolas worldlines: Rindler spacetime**

Now more interesting is the variant where the worldlines are embedded hyperbola having same asymptotes [$x = \mathrm{sqrt}(t^2+k^2)$]

Here the acceleration is constant in each co-moving frame but different. Chasing rocket has a greater acceleration than chased rocket but never catch it. Embedded hyperbolas $x = \mathrm{sqrt}(t^2+k^2)$ never intersect.
Here, surprisingly as acceleration are different, distance between rockets remain constant (we decide this as there is no relative celerity of each rocket in the (common) local frame of the other.

How is this possible?. This is possible because of the curvature of the spacetime.

The foliation of the 2D Minkowski space according to the coordinates $(\eta, \xi)$ of the Rindler spacetime (see chapter 7.3) is such as time sections (at spacelike coordinate $\xi = cste$) of this spacetime (a *1D* timelike "hypersurface" i.e a line) are orthogonal to the space basis vector $(\partial_\xi)^\mu$.
This is confirmed by the diagonal form of the metric in these coordinates.

This timelike coordinate (at $\xi = cste$) is proportional to the proper time of an uniformly accelerated observer with acceleration $\alpha = a.e^{-a\xi.} = a.e^{-k}$ according to the equations and the condition we imposed in chapter 7.3 for getting the distance invariance between hyperbolas of the one parameter family of hyperbolas.
This proportionality is given by $\eta(\tau) = \alpha.\tau/a = e^{-a\xi.}\tau = e^{-k}\tau$

Therefore these coordinates provide a set of spatially "equidistant" hyperbolas", the affine parameter of which is proportional to proper time of uniformly accelerated observers, orthogonal to the (straight) lines of constant time, the affine parameter of which is the distance between hyperbolas, i.e the equivalent for accelerated observers of the Minkowski Cartesian coordinates for static observers.
We be wonder why we have to adjust the acceleration of the observers according to the rule we stated before.
A Wick rotation would make it easy to understand.
Let's remind that a set of such hyperbolas corresponds to a set of concentric circles under a Wick rotation (chapter9, fig 12bis).
With this in mind the solution is obvious. Any set of two concentric circles exhibit a constant distance between them. But the curvature of each circle (the inverse of the radius) is different. Let's notice that this

set of concentric circles generates the whole 2D plane and is nothing else that the $r$ = *constan*t lines in polar coordinates in euclidean geometry. Now going back to the cinematic, in euclidean geometry, whether we consider the curvature as the worldline of an observer, he would be uniformly accelerated and acceleration corresponds o the curvature[38].
So what is obvious in this property is conserved by a Wick rotation (see fig.12bis of chapter 9).

We see that the Cartesian diagram in euclidean geometry is transformed into Cartesian diagram in Minkowski space by a Wick rotation and polar coordinates are transformed into hyperbolic polar coordinates in the Minkwoski spacetime.

The Rindler $(\eta, \xi)$ coordinates are nothing else than polar coordinates in Minkowski spacetime!

We demonstrated these property in a 2D spacetime but it could be easily extended to a 4D Minkowski spacetime $(t, x, y, z)$ where acceleration would be in the $z$ direction. As this does not involve additional phenomenology we will not consider this case here which is beyond our scope (see appendix 8).

Again, deceleration would imply some anti symmetrical effect and it is possible as in the previous case to perform a round trip worldline by completing a convenient sequence of acceleration and deceleration steps. Again the twin paradox will be enjoyed by travelers.

Can we build something equivalent to proper time for the spatial proper distance?
Obviously we can integrate the value of the “proper” length all along the worldline (appendix 9).

**Minkowski space time in null coordinates**

This representation may enlighten some fundamental properties of this spacetime by exhibiting the structural part of light in SR (and GR). This reminding the importance of the Newmann Penrose Formalism in GR (Kerr Newmann BH).

To be written.

**What happens to an elastic cable?**

Why should the length increase under constant acceleration?

Isn't this equivalent to applying a constant force pulling both ends of the elastic cable?
In case, the length should increase provoking a backward elasticity reaction force in order to balance the acceleration and whether the acceleration is within elasticity limit (let say *1g*), equilibrium may be reached and the length should remain constant.
As far as we know the elastic stretch does depend on the force applied but does not on application time.

And whether we go back to inertial flight, there would be no longer a constraint on the cable so whether we are in the elasticity limit shouldn't the cable recover its initial length?

What would happen whether the cable is tight to the rocket but encounter the same acceleration applied on each of its atoms? Would it be stretched? Likely yes because of the curvature of spatial dimension while accelerating and at the end will it be frozen at its new size or will it recover its initial size.

And what happens whether at the end of acceleration, in inertial flight, in order to erase its “material memory”, we melt the cable and rebuilt from it a brand new one in these new frame (crystals of the steel would grow in this new context) of the same diameter (which was not subject to Lorentz contraction in

38 We did not demonstrate this well know and basic property , you may exercise it is straightforward.

this experience). Which length would it get. Would be its length equal to the current distance between rockets or would it be too short (original length)?

Wouldn't consistency with SR imply that its length will fit the current distance between ship's?

**Conclusion**

This paradox is more confusing that it appears and raise a fundamental problem about the Minkowski spacetime. There is no preferred frame in this space time but relative time and space elasticity looks to be a physical property. Even when acceleration, so stress ceases the space time parameters looks to stick to the value they acquired by the acceleration as this acceleration was just a transient media to make a body come to a frame where physical properties of space and time are different, like what happens when we move in a gravitational field (but in a gravitational field we have a physical motivation) . Even though we have seen that this does not imply any contradiction there is a conceptual difficulty. Within a classical interpretation, it difficult to understand how such property is physically possible.
Paradoxically SR is more strange and difficult to understand than GR!
We should suppose that classical interpretation is not possible and that the structure of the Minkowski space whether it exists physically (we may doubt of this, it may be some asymptotic limit of mass system when we are very far from the masses) should be the right one even if this is out of the scope of classical understanding.

# 11- Appendixes

## 11-1 Appendix 1: length in Minkowski metric vs euclidean metric

As the Minkowski diagram is represented in euclidean geometry, the length of the curve on the diagram is $\int(dx^2+dt^2)^{1/2}$, in the Minkowski geometry this length is $\int(dx^2-dt^2)^{1/2}$,which is different!

## 11-2 Appendix 2: Equation of curves in maxima

Maxima 5.9.2 http://maxima.sourceforge.net
Using Lisp GNU Common Lisp (GCL) GCL 2.6.7 (aka GCL)
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
This is a development version of Maxima.

## 11-3 Appendix 3:$d^2x/dt^2$ =1 at t =0

According to:

$$v = dx/dt \text{ at } t = t_0$$

and

$$x = sqrt(1+t^2) \rightarrow v = dx/dt = t/sqrt(1+t^2)$$

$$d^2x/dt^2 = d/dt(dx/dt) = d/dt(t/sqrt(1+t^2) = 1/sqrt(1+t^2) - t.\{[t/[sqrt(1+t^2).(1+t^2)]\}= (1+t^2 -t^2)(1+t^2)^{-3/2}$$

$$d^2x/dt^2 = (1+t^2)^{-3/2} = 1 \text{ for } t = 0$$

## 11-4 Appendix 4 : $d^2x'/dt'^2 = 1$ in basic solution: $x =sqrt(1+t^2)$

The Minkowski local co-moving frame at $P$ is defined by its constant velocity $v$ relative to the laboratory frame (so the $\gamma$ factor is also constant).The value of this velocity depending on the point $P$ on the worldline should be plugged, as well as $\gamma$, in the equations after having computed first and second derivatives.

According to:

$t' = \gamma(t-vx) + C\_0 \qquad v = dx/dt$ at $t = t_0$

$x' = \gamma(x-vt) + D\_0 \qquad \gamma = 1/sqrt(1-v^2)$

and

$x = sqrt(1+t^2) \rightarrow v = dx/dt = t/sqrt(1+t^2) \rightarrow dx = t/sqrt(1+t^2).dt$

$so\ \gamma = 1/sqrt[1- (t^2/1+t^2)] = 1/sqrt\ (1/1+t^2) = sqrt(1+t^2)$

at some point $P$ at coordinates $t = t_0$,

$dt' = \gamma[1- v.t/sqrt(1+t^2)]dt = \gamma[(sqrt(1+t^2)- v.t)/sqrt(1+t^2)]dt$

$dx' = \gamma[\ t/sqrt(1+t^2) - v]dt = \gamma[\ (t - sqrt(1+t^2))/sqrt(1+t^2)]dt$

So velocity in the co-moving frame is:

$dx'/dt' = [\ t/sqrt(1+t^2) - v]/[1- v.t/sqrt(1+t^2)] = [t -v.sqrt(1+t^2)]/[sqrt(1+t^2) - v.t]$

Acceleration in co-moving frame $d^2x'/dt'^2 = d/dt'(dx'/dt) = d/dt(dx'/dt')(dt/dt') =$

$\{d/dt([t -v.sqrt(1+t^2)]/[sqrt(1+t^2) - v.t])\}\{sqrt(1+t^2)/[sqrt(1+t^2) -v.t]\}/\gamma =$

$\{[1 -v.t/sqrt(1+t^2)]/[sqrt(1+t^2)-v.t] - ([t-v.sqrt(1+t^2)]/[sqrt(1+t^2)-v.t]^2)([t-v.sqrt(1+t^2)]/[sqrt(1+t^2)])\}$
$\{sqrt(1+t^2)/[sqrt(1+t^2) -v.t]\}/\gamma =$

$\{1/sqrt(1+t^2)\}\{[sqrt(1+t^2) - v.t]/[sqrt(1+t^2)-v.t]- [t-v.sqrt(1+t^2)][t-v.sqrt(1+t^2)]/sqrt(1+t^2) -v.t)^2\}$
$\{sqrt(1+t^2)/[sqrt(1+t^2) -v.t]\}/\gamma$

Many terms cancel and this can be simplified in:

$\{([sqrt(1+t^2) -v.t]^2- [t-v.sqrt(1+t^2)]^2)/[sqrt(1+t^2) -v.t]^3\}/\gamma = \{[sqrt(1+t^2)-v.t +t-v.sqrt(1+t^2)][sqrt(1+t^2]-vt$
$-t +v.sqrt(1+t^2)]\}/[sqrt(1+t^2) -v.t]^3\}/\gamma =$
$\{[(1-v)[sqrt(1+t^2)+t](1+v)[sqrt(1+t^2)-t]\}\{[sqrt(1+t^2) -v.t]^3\}/\gamma = \{[(1-v^2)(1+t^2-t^2))]/[sqrt(1+t^2) -v.t]^3\}/\gamma =$

$[(1-v^2)^{3/2}]/[sqrt(1+t^2) -v.t]^3\}$

as $\gamma = 1/\ sqrt(1-v^2) = sqrt(1+t^2)$ and $v = t/sqrt(1+t^2)$, plugging these into previous equation gives:

$[(1-v^2)^{3/2}][sqrt(1+t^2) -v.t]^3\} = \{[sqrt[(1/1+t^2)]]/[sqrt(1+t^2)-t^2/sqrt(1+t^2)]\}^3 =$

$\{[sqrt[(1/1+t^2)]]/[(1+t^2-t^2)/sqrt(1+t^2)\}^3 = \{sqrt[(1/1+t^2)sqrt(1+t^2)\}^3 = 1.$

So we see that the acceleration in co-moving frame is constant and is equal to *1,* as asserted.

The same computation applies for rocket *2* complying to equation:

$x = sqrt(1+t^2) + K$

as $v = dx/dt = t/sqrt(1+t^2)$ identical to first case, all differential elements have the same value than in the first case (the constant K is eliminated).

# 11-5 Appendix 5 : velocity of rocket 2 in rocket 1 co-moving frame.

In $t', x'$ coordinates at $P$, $dx'/dt' = 0$ as $t'$- axis is tangent to rocket 1 worldline.

To compute $dx'/dt'$ at Q in t', x' coordinates, we have to compute the intersection of x'-axis with rocket 2 worldline for getting the coordinates of point $Q$ and to compute the equation of the tangent of worldline of rocket 2 at Q.
By comparing the slope of the tangent straight line at P and Q we may check that dx'/dt' > 0 at Q, which is obvious on fig.2.
According to our example on fig.2, at $P$ of coordinates $t = 0.5, x = sqrt(1.25)$ in lab frame, we get:

x = sqrt(1.25)*2t.

This line crosses rocket 2 worldline at $Q$, of coordinates $t, x$ in the lab frame defined by:

$x = sqrt(1.25)*2t = sqrt(t^2+1) + 0.5 \rightarrow (sqrt(1.25)*2t - 0.5)^2 = t^2+1 \rightarrow 5t^2 + 0.25 - 2*sqrt(1.25)t = t^2+1 \rightarrow$

$4t^2 - 2*sqrt(1.25) - 0.75 = 0$, selecting positive root:

$t = [sqrt(1.25) + sqrt(1.25+3)]/4 \approx 0.7948967 \approx 0.7949$
$x = sqrt(1.25)*2*0.7948967$.

The tangent at rocket 2 worldline at $Q$ ($t = 0.7949$) is given by:
$dx/dt_{(Q)} = t/sqrt(1+t^2) = 0.7949/sqrt[1+(0.7949)^2] \approx 0.62226$

While the tangent at rocket 1 worldline at $P$ ($t = 0.5$), which is the t'-axis of co-moving frame was:

$dx/dt_{(P)} = t/sqrt(1+t^2) = 0.5/sqrt[(0.5)^2 + 1] \approx 0.4472$

we see that the slope of the tangent at $Q$ is higher than at $P$, so the velocity of rocket 2 at $Q$ is positive in rocket 1 co -moving frame.

# 11-6 Appendix 6 : $d^2x'/dt'^2 = 1$ in variant solution: $x = sqrt(k^2+t^2)$

As this demonstration is very close from these of appendix 4 we follow the same method
According to:

$$t' = \gamma(t-vx) + C\_0 \qquad v = dx/dt \text{ at } t = t_0$$
$$x' = \gamma(x-vt) + D\_0 \qquad \gamma = 1/sqrt(1-v^2)$$

and

$$x = sqrt(k^2+t^2) \rightarrow v = dx/dt = t/sqrt(k^2+t^2) \rightarrow dx = t/sqrt(k^2+t^2)dt$$
$$\text{so } \gamma = 1/sqrt[1-(t^2/k^2+t^2)] = 1/sqrt(k^2/k^2+t^2) = sqrt(1+t^2/k^2)$$

at some point $P$ at coordinates $t = t_0$ ,

$$dt' = \gamma[1- v.t/sqrt(k^2+t^2)]dt = \gamma\{[sqrt(k^2+t^2) - v.t]/sqrt(k^2+t^2)\}dt$$
$$dx' = \gamma[t/sqrt(1+k^2) - v]dt = \gamma\{[t - v.sqrt(k^2+t^2)]/sqrt(1+k^2)\}dt$$

So velocity in the co-moving frame is:

$$dx'/dt' = [t/sqrt(k^2+t^2) - v]/[1- v.t/sqrt(k^2+t^2)] = [t - v.sqrt(k^2+t^2)]/[sqrt(k^2+t^2) - v.t]$$

Acceleration in co-moving frame $d^2x'/dt'^2 = d/dt'(dx'/dt) = d/dt(dx'/dt')(dt/dt') =$

$$\{d/dt([t - v.sqrt(k^2+t^2)]/[sqrt(k^2+t^2) - v.t])\}\{sqrt(k^2+t^2)/[sqrt(k^2+t^2) - v.t]\}/\gamma =$$

$$\{[1 - v.t/sqrt(k^2+t^2)]/[sqrt(k^2+t^2)-v.t] - ([t-v.sqrt(k^2+t^2)]/[sqrt(k^2+t^2)-v.t]^2)([t-v.sqrt(k^2+t^2)]/[sqrt(k^2+t^2)])\}$$
$$\{sqrt(k^2+t^2)/[sqrt(k^2+t^2) - v.t]\}/\gamma =$$

$\{1/\mathrm{sqrt}(k^2+t^2)\}\{[\mathrm{sqrt}(k^2+t^2) - v.t]/[\mathrm{sqrt}(k^2+t^2)-v.t]- [t-v.\mathrm{sqrt}(k^2+t^2)][t-v.\mathrm{sqrt}(k^2+t^2)]/\mathrm{sqrt}(k^2+t^2) -v.t)^2\}$
$\{\mathrm{sqrt}(k^2+t^2)/[\mathrm{sqrt}(k^2+t^2) -v.t]\}/\gamma$

Many terms cancel and this can be simplified in:

$\{([\mathrm{sqrt}(k^2+t^2) -v.t]^2- [t-v.\mathrm{sqrt}(k^2+t^2)]^2)/[\mathrm{sqrt}(k^2+t^2) -v.t]^3\}/\gamma =\{[\mathrm{sqrt}(k^2+t^2)-v.t +t-v.\mathrm{sqrt}(k^2+t^2)][\mathrm{sqrt}(k^2+t^2]-$
$vt -t +v.\mathrm{sqrt}(k^2+t^2)]\}/[\mathrm{sqrt}(k^2+t^2) -v.t]^3\}/\gamma =$

$[\{[(1-v)[\mathrm{sqrt}(k^2+t^2)+t](1+v)[\mathrm{sqrt}(k^2+t^2)-t]\}/\{[\mathrm{sqrt}(k^2+t^2) -v.t]^3\}]/\gamma =\{[(1-v^2)(1+t^2-t^2))]/[\mathrm{sqrt}(k^2+t^2)$
$-v.t]^3\}/\gamma = [(1-v^2)^{3/2}]/[\mathrm{sqrt}(k^2+t^2) -v.t]^3\}$

as $\gamma = 1/ \mathrm{sqrt}(1-v^2) = \mathrm{sqrt}(k^2+t^2)/k$ and $v = t/\mathrm{sqrt}(k^2+t^2)$, plugging these into previous equation gives:

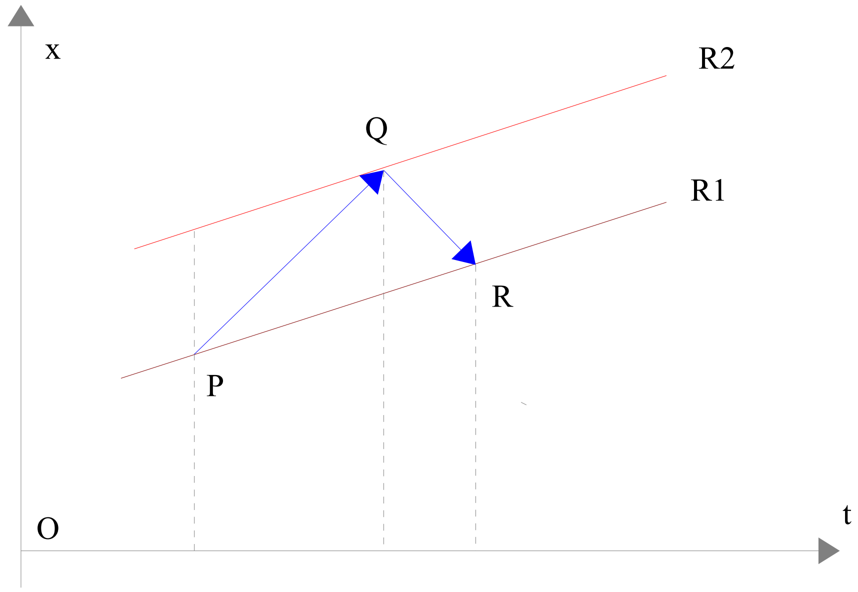
$[(1-v^2)^{3/2}]/[\mathrm{sqrt}(k^2+t^2) -v.t]^3\}= \{[\mathrm{sqrt}[(k^2/k^2+t^2)]]/[\mathrm{sqrt}(k^2+t^2)-t^2/\mathrm{sqrt}(k^2+t^2)]\}^3=$
$\{[\mathrm{sqrt}[(k^2/k^2+t^2)]]/[(k^2+t^2-t^2)/\mathrm{sqrt}(k^2+t^2)\}^3= \{\mathrm{sqrt}[(k^2/k^2+t^2)\mathrm{sqrt}(k^2+t^2)/k^2\}^3= k^{-3/2}$.

So we see that the acceleration of the pursued rocket in co-moving frame is constant as asserted and is equal to $k^{-3/2}, < 1$ as $k > 1$ , which is inferior to the acceleration of the pursuing rocket.

# 11-7 Appendix 7: Direct computation of proper distance in inertial flight

## 11-7-1- Distance between rockets in rocket's co-moving frame

We will use the radar method. From *P* $(t_0,x_0)$ belonging to *R1* (rocket 1 inertial worldline) a light signal is emitted, it reaches *R2* (rocket 2 inertial worldline ) at Q $(t_1,x_1)$ where it is reflected and reaches *R1* at R$(t_2,x_2)$. *R1* and *R2* worldlines are separated by a distance *d* in the lab frame and have velocity *v* in lab frame.




Equation of *R1* (brown line) is: $x = v.t + a$

as *P* belongs to *R1* we have $x_0 = v.t_0 + a \rightarrow a = x_0 - vt_0, \rightarrow x = vt + x_0 - vt_0$

Equation of *R2* (*R1* translated, red line) is obviously: $x = vt + x_0 + d - vt_0$

Equation of outgoing light ray (with $c = 1$) worldline *OL (blue)* at *P* is: $x = t + b$

as *P* belongs to *R1* we have: $x_0 = t_0 + b \rightarrow b = x_0 - t_0, \rightarrow x = t + x_0 - t_0$

*OL* crosses R2 at *Q* $(t_1,x_1)$ defined by:

$t + x_0 - t_0 = vt + x_0 + d - vt_0 \rightarrow$

$t(1-v) = t_0(1-v) + d \rightarrow t_1 = t_0 + d/(1-v)$

$x_1 = d/(1-v) + x_0$

Equation of reflected light ray *RL* (blue) at *Q* is: $x = -t + c$

as *Q* belongs to RL we have:

$d/(1-v) + x_0 = -[t_0 + d/(1-v)] + c \rightarrow c = 2d/(1-v) + x_0 + t_0 \rightarrow x = -t + 2d/(1-v) + x_0 + t_0$

*RL* crosses *R1* at *R*$(t_2,x_2)$ defined by:

$$vt + x_0 - vt_0 = -t + 2d/(1-v) + x_0 + t_0 \rightarrow t(1+v) = t_0(1+v) + 2d/(1-v) \rightarrow t = t_0 + 2d/(1-v^2)$$

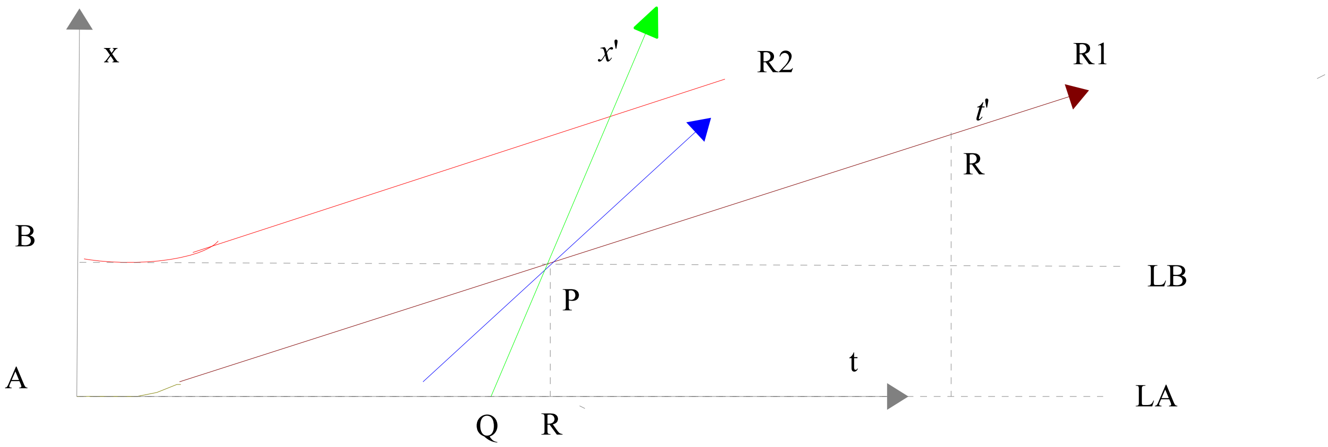
The proper time on segment *PR* of *R1* is:

$\tau^2 = t^2 - x^2 = t^2 - v^2t^2 = t^2(1-v^2) = [t_0 + 2d/1-v^2) - t_0]^2[1 - v^2] \rightarrow \tau = 2d/sqrt(1-v^2),$

as in the radar method, this is twice the distance *D* (return path) multiplied by speed of light, we have:

$$D = d/sqrt(1-v^2)$$

# 11-7-2- Initial distance in lab frame measured in co-moving frame.




The worldlines of rockets 1 and two are sketched (with their non linear part) on the diagram.
*A* and *B* are the taking off places of rockets 1 and 2 in lab frame. We would like to measure in *R1* inertial part of the worldline the distance *AB*. Let's call it *d* as usual. *P* has coordinates ($t_0$, $d$)
We also drew co-moving R1 frame *t'*-axis (which is R1 worldline) and *x'*-axis which is symmetrical of *t'*-axis around a light ray (*eq.* $x = t$ ) issued at *P*. So whether slope of *t'*-axis is *v*, slope of *x'*-axis would be *1/v*.
We know that x'-axis is the line of simultaneity.
PQ is the distance d would be measured in R1 frame.

The generic equation of x'-axis in lab frame is:

$$x' = t/v + a$$

*P* is on the line so :

$$d = t_0/v + a \rightarrow x' = t/v + d - t_0/v$$

This line crosses x-axis for $x' = 0$, i.e :

$$t/v + d - t_0/v = 0 \rightarrow t = t_0 - v.d$$

The proper distance *D* (*PQ* on the diagram) in R1 co-moving frame in *SR* is therefore:

$$s^2 = -t^2 + x^2 = -(QR)^2 + d^2 = -(v.d)^2 + d^2 = d^2\,(1-v^2) \rightarrow s = d.\,(1-v^2)^{1/2} \quad .$$

Which is what we expected: distance *d* looks contracted when measured in co-moving frame.

# 11-8 Appendix 8: Length integration on worldlines

We have seen that the length is different for accelerating rocket 1 and accelerating rocket 2.
Let's first compute the length for rocket 1.

## 11-9 Appendix 9: 4D Rindler spacetime.

Cylindrical coordinates ?

$$ds^2 = e^{2\varphi} d\bar{t}^2 - e^{-2\varphi}\left[e^{2\sigma}\left(d\bar{r}^2 + d\bar{z}^2\right) + \bar{r}^2 d\theta^2\right] \qquad (1)$$

## 12- References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## 10 Further reading

See also

- Ehrenfest paradox
- Physical paradox
- Supplee's paradox
- Rindler coordinates
- Twin paradox
- Born rigidity
- Hyperbolicmotion (relativity)


- Michael Weiss, Bell's Spaceship Paradox (1995), USENET Relativity FAQ
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